Correlation functions of an interacting spinless fermion model at finite temperature

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Abstract

We formulate correlation functions for a one-dimensional interacting spinless fermion model at finite temperature. By combination of a lattice path integral formulation for thermodynamics with the algebraic Bethe ansatz for fermion systems, the equal-time one-particle Green's function at arbitrary particle density is expressed as a multiple integral form. Our formula reproduces previously known results in the following three limits: the zero-temperature, the infinite-temperature and the free fermion limits.

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1 Introduction

The exact computation of correlation functions for strongly correlated quantum systems has been one of the major problems for years. Although, in general, this is exceedingly difficult to achieve, several analytical approaches especially in 1D quantum integrable models have been provided to derive exact or manageable expressions of correlation functions. For instance, the low-energy behavior of correlation functions for gapless models can be systematically obtained by conformal field theory [1–4]. On the other hand, for systems with finite spectral gaps, the long-distance and -time asymptotics are investigated by (finite-temperature) form factor expansions (see [5, 6] for recent developments).

An alternative approach, which has been developed in these several years particularly for the spin-1/2 XXZ chain, is to combine the algebraic Bethe ansatz [4] with solutions to the quantum inverse problem for local spin operators [7]. Using this, Kitanine *et al* derived multiple integral representations for zero-temperature correlation functions of the XXZ chain with an external field [8–10]. Their representations can be regarded as natural extensions

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of the results based on the q-vertex operator approach [11–13], which is restricted to the zero-magnetic field case. One of the advantages of this method is that the formulation can be flexibly generalized to finite temperature and/or time dependent case [14–17] by combining a lattice path integral formulation. Furthermore, by considering a continuum limit of the XXZ chain, correlation functions of the 1D boson system with delta function interaction can be obtained at finite temperature [18] (see also [19] for the zero-temperature case).

Beyond spin systems, more recently we further extended the method to the calculation of correlation functions for fermion systems. By use of the fermionic R-operator [20] acting directly on the fermionic Fock space, we have derived multiple integral representations of zero-temperature correlation functions for an interacting spinless fermion model with arbitrary particle density [21]. In this paper, we generalize the former results to the finite-temperature case by use of the quantum transfer matrix technique utilizing a concept of path integral [22]. Especially considered here is the equal-time one-particle Green's function. Our formula agrees with previously known results in the following three limits: the zero-temperature, infinite-temperature and the free fermion limits.

The layout of the paper is as follows. In the next section, we review the quantum transfer matrix method for the spinless fermion model, and express the correlation function in terms of matrix elements of the monodromy operator. In section 3, we present the key ingredients of the computation for the correlation function. The multiple integral representation for the equal-time one-particle Green's function at finite temperature with arbitrary particle density is summarized in the main theorem. In section 4, the three special limits are evaluated. Section 5 is devoted to a brief discussion. The detailed derivation of the multiple integral form is deferred to the appendix.

2 Spinless fermion model

In this section, the thermodynamics of an interacting spinless fermion model is formulated by the quantum transfer matrix method [22]. The two-point correlation functions at finite temperature are expressed in terms of matrix elements of the monodromy operator.

2.1 Fermionic R-operator

The Hamiltonian of the interacting spinless fermion model on a 1D periodic lattice with L sites is defined as

$$
\begin{aligned}
H &= H_0 - \mu_{\mathrm{c}} \sum_{j=1}^{L} \left(\frac{1}{2} - n_j \right), \\
H_0 &= t \sum_{j=1}^{L} \left\{ c_j^{\dagger} c_{j+1} + c_{j+1}^{\dagger} c_j + 2\Delta \left(\left(\frac{1}{2} - n_j \right) \left(\frac{1}{2} - n_{j+1} \right) - \frac{1}{4} \right) \right\},
\end{aligned} \tag{2.1}
$$

where $c_j^{\dagger}$ and c_j are the fermionic creation and annihilation operators at the jth site, respectively, satisfying the canonical anti-commutation relations. Here t and Δ are real constants characterizing the nature of the ground state, and μ_{c} denotes the chemical potential coupling to the density operator $n_j = c_j^{\dagger} c_j$.

The underlying integrability of the model (2.1) can be seen by introducing the fermionic R-operator defined as

$$R_{\bar{i}j}(\lambda) = 1 - n_{\bar{i}} - n_j + \frac{\operatorname{sh}\lambda}{\operatorname{sh}(\lambda+\eta)}(n_{\bar{i}} + n_j - 2n_{\bar{i}}n_j) + \frac{\operatorname{sh}\eta}{\operatorname{sh}(\lambda+\eta)}(c_{\bar{i}}^\dagger c_j + c_j^\dagger c_{\bar{i}}), \qquad (2.2)$$

which acts on $V_{\bar{i}} \otimes_{\mathrm{s}} V_j$. Here V_k is a two-dimensional fermion Fock space whose normalized orthogonal basis is given by $|0\rangle_k$ and $|1\rangle_k := c_k^\dagger|0\rangle_k$, where $c_k|0\rangle_k = 0$ and $\otimes_{\mathrm{s}}$ denotes the super tensor product. Identifying the above Fock space V_j (respectively $V_{\bar{i}}$) with the quantum space $\mathcal{H}_j$ (respectively the auxiliary space $\mathcal{H}_{\bar{i}}$), we define the monodromy operator $\mathcal{T}_{\bar{i}}^{\mathrm{R}}(\lambda)$ acting on the space $V_{\bar{i}} \otimes_{\mathrm{s}} (V_1 \otimes_{\mathrm{s}} V_2 \otimes_{\mathrm{s}} \ldots \otimes_{\mathrm{s}} V_L)$ as

$$\mathcal{T}_{\bar{i}}^{\mathrm{R}}(\lambda) = R_{\bar{i}L}(\lambda)\ldots R_{\bar{i}2}(\lambda)R_{\bar{i}1}(\lambda).$$

Since the fermionic R-operator (2.2) satisfies the Yang-Baxter equation [20]

$$R_{12}(\lambda_1-\lambda_2)R_{13}(\lambda_1)R_{23}(\lambda_2) = R_{23}(\lambda_2)R_{13}(\lambda_1)R_{12}(\lambda_1-\lambda_2),$$

the transfer operator

$$T_{\mathrm{R}}(\lambda) = \mathrm{Str}_{\bar{i}}\,\mathcal{T}_{\bar{i}}^{\mathrm{R}}(\lambda) = {}_{\bar{i}}\langle 0|\mathcal{T}^{\mathrm{R}}(\lambda)|0\rangle_{\bar{i}} - {}_{\bar{i}}\langle 1|\mathcal{T}^{\mathrm{R}}(\lambda)|1\rangle_{\bar{i}}$$

constitutes a commuting family: $[T_{\mathrm{R}}(\lambda), T_{\mathrm{R}}(\mu)] = 0$, where the dual fermion Fock space is spanned by ${}_k\langle 0|$ and ${}_k\langle 1|$ with ${}_k\langle 1| := {}_k\langle 0|c_k$ and ${}_k\langle 0|c_k^\dagger = 0$. The Hamiltonian H_0 (2.1) is expressed in terms of the logarithmic derivative of the transfer operator $T_{\mathrm{R}}(\lambda)$:

$$H_0 = t\operatorname{sh}(\eta)\frac{\partial}{\partial\lambda}\ln T_{\mathrm{R}}(\lambda)\bigg|_{\lambda=0}, \qquad \Delta = \operatorname{ch}\eta.$$

This relation yields

$$T_{\mathrm{R}}(\lambda) = T_{\mathrm{R}}(0)\left(1 + \frac{\lambda}{t\operatorname{sh}\eta}H_0 + \mathcal{O}(\lambda^2)\right). \qquad (2.3)$$

For later use, let us define another type of transfer operator $\overline{T}_{\mathrm{R}}(\lambda)$ [22]:

$$\overline{T}_{\mathrm{R}}(\lambda) = \mathrm{Str}_{\bar{i}}[\overline{R}_{\bar{i}L}(-\lambda)\ldots\overline{R}_{\bar{i}2}(-\lambda)\overline{R}_{\bar{i}1}(-\lambda)]$$

with

$$\overline{R}_{\bar{i}j}(\lambda) = R_{j\bar{i}}^{\mathrm{st}_j}(\lambda) = 1 - n_{\bar{i}} - n_j + \frac{\operatorname{sh}\lambda}{\operatorname{sh}(\lambda+\eta)}(n_{\bar{i}} + n_j - 2n_{\bar{i}}n_j) - \frac{\operatorname{sh}\eta}{\operatorname{sh}(\lambda+\eta)}(c_{\bar{i}}^\dagger c_j^\dagger - c_j c_{\bar{i}}), \quad (2.4)$$

where st_j denotes the supertranspose with respect to the jth space. Note that $T_{\mathrm{R}}(0)$ ($\overline{T}_{\mathrm{R}}(0)$) is the right-shift (left-shift) operator, namely $T_{\mathrm{R}}(0)x_j = x_{j+1}T_{\mathrm{R}}(0)$ ($\overline{T}_{\mathrm{R}}(0)x_j = x_{j-1}\overline{T}_{\mathrm{R}}(0)$) where $x_j = c_j, c_j^\dagger$, and hence $T_{\mathrm{R}}^{-1}(0) = \overline{T}_{\mathrm{R}}(0)$. Using this together with the expansion (2.3), one finds the statistical operator $\mathrm{e}^{-H/T}$ (T: temperature) is given by

$$\mathrm{e}^{-H/T} = \mathrm{e}^{\sum_{j=1}^{L}\mu_{\mathrm{c}}(1-2n_j)/(2T)}\lim_{N\to\infty}\left[\overline{T}_{\mathrm{R}}(\lambda)T_{\mathrm{R}}\left(\lambda - \frac{\beta}{N}\right)\right]^{\frac{N}{2}}\Bigg|_{\lambda=0}, \qquad \beta = \frac{2t\operatorname{sh}\eta}{T}, \qquad (2.5)$$

where the Trotter number N is assumed to be $N \in 2\mathbb{N}$. Note here that we have set the Boltzmann constant to unity.

2.2 Correlation functions at finite temperature

To derive multiple integral representations of finite temperature correlation functions, here we describe how the correlation functions can be expressed in terms of the transfer operator formalism.

Let us consider a two-point correlation function at finite temperature $T > 0$:

$$\langle \mathcal{O}_{m+1} \mathcal{O}_1^\dagger \rangle = \frac{\mathrm{Tr}\{\mathrm{e}^{-H/T} \mathcal{O}_{m+1} \mathcal{O}_1^\dagger\}}{\mathrm{Tr}\, \mathrm{e}^{-H/T}} \quad (m \geq 1), \tag{2.6}$$

where $\mathcal{O}_j$ is a local fermion operator. Inserting the formula (2.5) into the above, and using the fact that the R-operator (2.2) or (2.4) is Grassmann even, one obtains

$$\langle \mathcal{O}_{m+1} \mathcal{O}_1^\dagger \rangle = \lim_{N\to\infty} \frac{\mathrm{Str}_{\overline{1},\ldots,\overline{N}}\, \mathrm{Tr}_{1,\ldots,L}\{\mathcal{T}_L(0) \cdots (\mathcal{T}_{m+1}(0)\mathcal{O}_{m+1})\mathcal{T}_m(0) \cdots (\mathcal{T}_1(0)\mathcal{O}_1^\dagger)\}}{\mathrm{Str}_{\overline{1},\ldots,\overline{N}}\, \mathrm{Tr}_{1,\ldots,L}\{\mathcal{T}_L(0) \cdots \mathcal{T}_1(0)\}},$$

where the operator $\mathcal{T}_j(\lambda)$ acting in the space $(V_{\overline{1}} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} V_{\overline{N}}) \otimes_{\mathrm{s}} V_j$ is defined by

$$\begin{aligned}\mathcal{T}_j(\lambda) &= \mathrm{e}^{\mu_{\mathrm{c}}(1-2n_j)/(2T)} \overline{R}_{\overline{N}j}(-\lambda) R_{\overline{N-1}j}\left(\lambda - \frac{\beta}{N}\right) \cdots \overline{R}_{\overline{2}j}(-\lambda) R_{\overline{1}j}\left(\lambda - \frac{\beta}{N}\right) \\ &= A(\lambda)(1-n_j) + B(\lambda)c_j + c_j^\dagger C(\lambda) + D(\lambda)n_j.\end{aligned}$$

The Yang-Baxter equation and its modification

$$\overline{R}_{31}(-\lambda_2)\overline{R}_{32}(-\lambda_1)R_{12}(\lambda_1-\lambda_2) = R_{12}(\lambda_1-\lambda_2)\overline{R}_{32}(-\lambda_1)\overline{R}_{31}(-\lambda_2)$$

yield

$$\mathcal{T}_1(\lambda_2)\mathcal{T}_2(\lambda_1)R_{12}(\lambda_1-\lambda_2) = R_{12}(\lambda_1-\lambda_2)\mathcal{T}_2(\lambda_1)\mathcal{T}_1(\lambda_2),$$

and therefore the quantum transfer matrix defined by

$$T(\lambda) = \mathrm{Tr}_j\, \mathcal{T}_j(\lambda) = A(\lambda) + D(\lambda) \tag{2.7}$$

commutes for different spectral parameters: $[T(\lambda), T(\mu)] = 0$. Thus (2.6) reduces to

$$\langle \mathcal{O}_{m+1} \mathcal{O}_1^\dagger \rangle = \lim_{N\to\infty} \frac{\mathrm{Str}_{\overline{1},\ldots,\overline{N}}\{T^{L-m-1}(0)\, \mathrm{Tr}_{m+1}\{\mathcal{T}_{m+1}(0)\mathcal{O}_{m+1}\} T^{m-1}(0)\, \mathrm{Tr}_1\{\mathcal{T}_1(0)\mathcal{O}_1^\dagger\}\}}{\mathrm{Str}_{\overline{1},\ldots,\overline{N}}\, T^L(0)}.$$

Let us consider the thermodynamic limit $L \to \infty$. Since the two limits $L \to \infty$ and $N \to \infty$ are interchangeable [23, 24], we can take the limit $L \to \infty$ first. In addition, we find that the leading eigenvalue of the quantum transfer matrix $T(0)$ (written as $\Lambda_0(0)$) is non-degenerate and separated from the next-leading eigenvalues by a finite gap even in the Trotter limit $N \to \infty$. In the thermodynamic limit $L \to \infty$, therefore, (2.6) can be written in terms of $\Lambda_0(0)$ and the corresponding (normalized) eigenstate $|\Psi_0\rangle$ (note that $\Lambda_0(\lambda) := \langle \Psi_0 | T(\lambda) | \Psi_0 \rangle$). Namely

$$\langle \mathcal{O}_{m+1} \mathcal{O}_1^\dagger \rangle = \lim_{N\to\infty} \frac{\langle \Psi_0 |\, \mathrm{Tr}_{m+1}\{\mathcal{T}_{m+1}(0)\mathcal{O}_{m+1}\}(A+D)^{m-1}(0)\, \mathrm{Tr}_1\{\mathcal{T}_1(0)\mathcal{O}_1^\dagger\} | \Psi_0 \rangle}{\Lambda_0^{m+1}(0)}. \tag{2.8}$$

In particular, for the equal-time one-particle Green's function (set $\mathcal{O}_j = c_j$), which will mainly be considered in this paper, one obtains

$$\langle c_{m+1} c_1^\dagger \rangle = \langle c_1 c_{m+1}^\dagger \rangle = -\lim_{N\to\infty} \Lambda_0^{-m-1}(0) \langle \Psi_0 | C(0)(A+D)^{m-1}(0)B(0) | \Psi_0 \rangle. \tag{2.9}$$

Note that we have used $\mathrm{Tr}_j\{\mathcal{T}_j(0)c_j\} = -C(0)$ and $\mathrm{Tr}_j\{\mathcal{T}_j(0)c_j^\dagger\} = B(0)$.

2.3 Diagonalization of the quantum transfer matrix

To evaluate the correlation function (2.8) (or (2.9)) actually, one must investigate the leading eigenvalues $\Lambda_0(0)$ and the corresponding eigenstates $|\Psi_0\rangle$ of the quantum transfer matrix $T(0)$. In this subsection, we present a general formula describing the eigenvalues and the eigenstates of $T(\lambda)$. The leading eigenvalue $\Lambda_0(0)$ is expressed via the solution to a nonlinear integral equation.

Let us define the reference state $|\Omega\rangle$ as

$$|\Omega\rangle := |0\rangle_{\bar{1}} \underset{\mathrm{s}}{\otimes} |1\rangle_{\bar{2}} \underset{\mathrm{s}}{\otimes} \cdots |0\rangle_{\overline{N-1}} \underset{\mathrm{s}}{\otimes} |1\rangle_{\bar{N}}. \tag{2.10}$$

Obviously (2.10) is an eigenstate of $T(\lambda)$ (2.7):

$$T(\lambda) = (a(\lambda) + d(\lambda))|\Omega\rangle, \quad A(\lambda)|\Omega\rangle = a(\lambda)|\Omega\rangle, \quad D(\lambda)|\Omega\rangle = d(\lambda)|\Omega\rangle,$$

where

$$a(\lambda) = \left\{ \frac{\operatorname{sh}\lambda}{\operatorname{sh}(\lambda-\eta)} \right\}^{\frac{N}{2}} \mathrm{e}^{\frac{\mu_{\mathrm{c}}}{2T}}, \quad d(\lambda) = (-1)^{\frac{N}{2}} \left\{ \frac{\operatorname{sh}(\lambda - \frac{\beta}{N})}{\operatorname{sh}(\lambda - \frac{\beta}{N} + \eta)} \right\}^{\frac{N}{2}} \mathrm{e}^{-\frac{\mu_{\mathrm{c}}}{2T}}.$$

In the framework of the algebraic Bethe ansatz, the vector $|\{\lambda\}\rangle$ constructed by the multiple action of $B(\lambda)$ on $|\Omega\rangle$, namely $|\{\lambda\}\rangle = \prod_{j=1}^{M} B(\lambda_j)|\Omega\rangle$, is an eigenstate of $T(\lambda)$ if the complex parameters $\{\lambda_j\}_{j=1}^{M}$ satisfy the Bethe ansatz equation:

$$\frac{a(\lambda_j)}{d(\lambda_j)} = -(-1)^M \prod_{k=1}^{M} \frac{\operatorname{sh}(\lambda_j - \lambda_k + \eta)}{\operatorname{sh}(\lambda_j - \lambda_k - \eta)}. \tag{2.11}$$

The corresponding eigenvalue of $T(\lambda)$ is written as

$$\Lambda(\lambda) = a(\lambda) \prod_{j=1}^{M} \frac{\operatorname{sh}(\lambda - \lambda_j - \eta)}{\operatorname{sh}(\lambda - \lambda_j)} + (-1)^M d(\lambda) \prod_{j=1}^{M} \frac{\operatorname{sh}(\lambda - \lambda_j + \eta)}{\operatorname{sh}(\lambda - \lambda_j)}. \tag{2.12}$$

The Bethe roots $\{\lambda\}$ characterizing the leading eigenvalue $\Lambda_0(0)$ are given by solutions to the Bethe ansatz equation (2.11) in the sector $M = N/2$. Then the Bethe ansatz equation (2.11) and the eigenvalue formula (2.12) are exactly the same as the spin-1/2 XXZ chain given by the Jordan-Wigner transformation, and hence we can directly utilize the method as in [25–27], which makes the analysis possible even in the Trotter limit $N \to \infty$. Let us consider the following auxiliary function

$$\mathfrak{a}(\lambda) = (-1)^{\frac{N}{2}} \frac{d(\lambda)}{a(\lambda)} \prod_{k=1}^{N/2} \frac{\operatorname{sh}(\lambda - \lambda_k + \eta)}{\operatorname{sh}(\lambda - \lambda_k - \eta)}, \tag{2.13}$$

which associates the Bethe roots $\{\lambda_j\}_{j=1}^{N/2}$ with zeros of $1 + \mathfrak{a}(\lambda)$. To study the analytical properties of this function, we need to know the distribution of the Bethe roots describing the leading eigenvalue. It has been numerically verified for a wide range of Trotter numbers that the roots are distributed inside $\mathcal{C}$. For instance, at $\mu_{\mathrm{c}} = 0$, the Bethe roots for the critical (off-critical) regime are located on the real (imaginary) axis. Thus we can safely

assume the above features hold for any Trotter numbers, from which the analytical properties of the auxiliary function are determined. Consequently one sees $\mathfrak{a}(\lambda)$ satisfies the following nonlinear integral equation:

$$\ln \mathfrak{a}(\lambda) = -\frac{\mu_{\mathrm{c}}}{T} + \frac{N}{2} \ln \frac{\operatorname{sh}(\lambda+\eta)\operatorname{sh}(\lambda-\frac{\beta}{N})}{\operatorname{sh}(\lambda)\operatorname{sh}(\lambda-\frac{\beta}{N}+\eta)} - \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{\operatorname{sh}(2\eta)\ln(1+\mathfrak{a}(\omega))}{\operatorname{sh}(\lambda-\omega+\eta)\operatorname{sh}(\lambda-\omega-\eta)}. \tag{2.14}$$

Here the contour $\mathcal{C}$ is taken, for instance, as a rectangular contour whose edges are parallel to the real axis at $\pm\pi\mathrm{i}/2$ (respectively $\pm\eta/2$) and are parallel to the imaginary axis at $\pm\eta/2$ (respectively $\pm\infty$) for the off-critical regime $\Delta = \operatorname{ch}\eta > 1$ (respectively for the critical regime $0 \le \Delta = \operatorname{ch}\eta \le 1$) (see figure 1 for a pictorial definition). In (2.14) the Trotter limit $N \to \infty$ can be taken analytically:

$$\ln \mathfrak{a}(\lambda) = -\frac{\mu_{\mathrm{c}}}{T} - \frac{t\operatorname{sh}^2(\eta)}{T\operatorname{sh}(\lambda)\operatorname{sh}(\lambda+\eta)} - \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{\operatorname{sh}(2\eta)\ln(1+\mathfrak{a}(\omega))}{\operatorname{sh}(\lambda-\omega+\eta)\operatorname{sh}(\lambda-\omega-\eta)}. \tag{2.15}$$

For later convenience, we also introduce another auxiliary function $\overline{\mathfrak{a}}(\lambda) = 1/\mathfrak{a}(\lambda)$ satisfying the following nonlinear integral equation in the limits $N \to \infty$ [15]:

$$\ln \overline{\mathfrak{a}}(\lambda) = \frac{\mu_{\mathrm{c}}}{T} - \frac{t\operatorname{sh}^2(\eta)}{T\operatorname{sh}(\lambda)\operatorname{sh}(\lambda-\eta)} + \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{\operatorname{sh}(2\eta)\ln(1+\overline{\mathfrak{a}}(\omega))}{\operatorname{sh}(\lambda-\omega+\eta)\operatorname{sh}(\lambda-\omega-\eta)}. \tag{2.16}$$

By the above auxiliary function $\mathfrak{a}(\lambda)$, the leading eigenvalue $\Lambda_0(0)$ of the quantum transfer matrix $T(0)$, which is related to the free energy density f by $f = -T\ln\Lambda_0(0)$, is expressed as the following single integral form:

$$\ln\Lambda_0(0) = \frac{\mu_{\mathrm{c}}}{2T} + \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{\operatorname{sh}(\eta)\ln(1+\mathfrak{a}(\omega))}{\operatorname{sh}(\omega)\operatorname{sh}(\omega+\eta)} = -\frac{\mu_{\mathrm{c}}}{2T} - \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{\operatorname{sh}(\eta)\ln(1+\overline{\mathfrak{a}}(\omega))}{\operatorname{sh}(\omega)\operatorname{sh}(\omega-\eta)}. \tag{2.17}$$

Differentiating (2.17) with respect to the chemical potential μ_{c}, one obtains the particle density $\langle n_j \rangle$

$$\langle n_j \rangle = -\int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{T\operatorname{sh}(\eta)\partial_{\mu_{\mathrm{c}}}\mathfrak{a}(\omega)}{\operatorname{sh}(\omega)\operatorname{sh}(\omega+\eta)(1+\mathfrak{a}(\omega))} = 1 + \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{T\operatorname{sh}(\eta)\partial_{\mu_{\mathrm{c}}}\overline{\mathfrak{a}}(\omega)}{\operatorname{sh}(\omega)\operatorname{sh}(\omega-\eta)(1+\overline{\mathfrak{a}}(\omega))}.$$

3 Multiple integral representation

Along the line developed in [15], we can derive a multiple integral representing the equal-time one-particle Green's function for the spinless fermion model. Here we sketch briefly how to derive the multiple integral by presenting some crucial formulae to evaluate the action of the operator $A + D$ and the resultant scalar product. These formulae are essentially the same with those for the zero-temperature case [21], since the commutation relations of the operators A, B, C and D are exactly the same with those for the zero-temperature case.

First, it is convenient to introduce the following more general function $\Phi_N(\{\xi\})$ instead of (2.9):

$$\begin{aligned}\Phi_N(\{\xi\}) &= -\frac{\langle\Psi_0|C(\xi_1)\prod_{j=2}^{m}(A+D)(\xi_j)B(\xi_{m+1})|\Psi_0\rangle}{\prod_{j=1}^{m+1}\Lambda_0(\xi_j)} \\ &= -\frac{\langle\{\lambda\}|C(\xi_1)\prod_{j=2}^{m}(A+D)(\xi_j)B(\xi_{m+1})|\{\lambda\}\rangle}{\langle\{\lambda\}|\{\lambda\}\rangle\prod_{j=1}^{m+1}\Lambda_0(\xi_j)},\end{aligned} \tag{3.1}$$

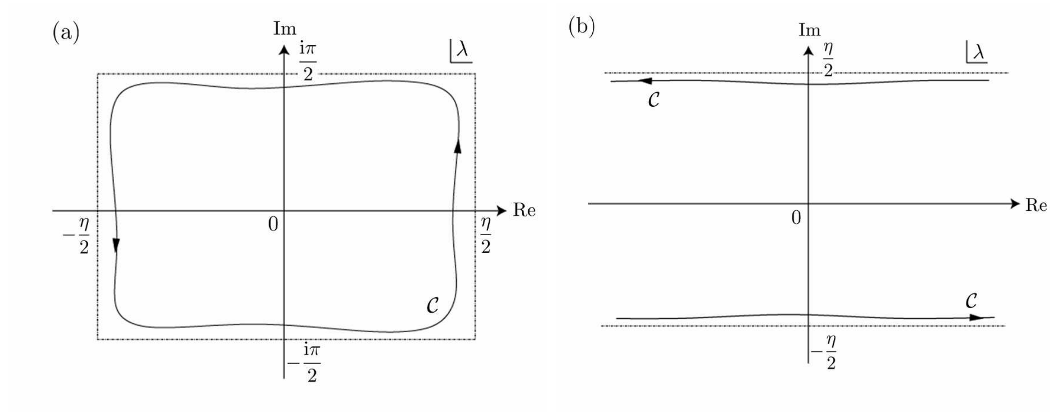

Figure 1: The integration contours for the off-critical regime $\Delta > 1$ (a) and for the critical regime $0 \leq \Delta \leq 1$ (b).

where $\{\xi_j\}_{j=1}^{m+1}$ is complex parameters located inside $\mathcal{C}$. Note that $\{\lambda\}$ and $|\{\lambda\}\rangle$ are, respectively, the Bethe roots and the eigenvector (not normalized), which characterize the leading eigenvalue $\Lambda_0(0)$ (see the preceding section). The dual vector $\langle\{\lambda\}|$ is constructed by the multiple action of $C(\lambda)$ on the state $\langle\Omega|$: $\langle\{\lambda\}| = \langle\Omega|\prod_{j=1}^{N/2} C(\lambda_j)$. It immediately follows that the one-particle Green's function (2.9) can be obtained by taking the homogeneous limit $\{\xi\} \to 0$ and the Trotter limit $N \to \infty$ in (3.1):

$$\langle c_1 c_{m+1}^\dagger \rangle = \lim_{N\to\infty} \lim_{\xi\to 0} \Phi_N(\{\xi\}). \tag{3.2}$$

To evaluate the multiple action of the operator $A + D$ on the state $\langle\{\lambda\}|C(\xi_1)$, let us introduce the following proposition, which is originally proposed in the calculation of the correlation function for the spin-1/2 XXZ chain [9].

Proposition 3.1 [21]. *The action of $\prod_{j=1}^m (A + \kappa D)(\xi_j)$ on a state $\langle\Omega|\prod_{j=1}^M C(\mu_j) = \langle\{\mu\}|$, for any sets of complex parameters $\{\mu_j\}_{j=1}^M$ (not necessarily the Bethe roots), is written as*

$$\langle\{\mu\}|\prod_{j=1}^m (A+\kappa D)(\xi_j) = \sum_{n=0}^{p} \sum_{\substack{\{\mu\}=\{\mu^+\}\cup\{\mu^-\} \\ \{\xi\}=\{\xi^+\}\cup\{\xi^-\} \\ |\mu^+|=|\xi^+|=n}} R_n(\{\xi^+\}|\{\xi^-\}|\{\mu^+\}|\{\mu^-\})\langle\{\xi^+\}\cup\{\mu^-\}|,$$

where $p = \min(m, M)$, $\langle\{\xi^+\}\cup\{\mu^-\}| = \langle\Omega|\prod_{j=1}^n C(\xi_j^+)\prod_{k=1}^{M-n} C(\mu_k^-)$ and the coefficient R_n is given by

$$\begin{aligned} R_n(\{\xi^+\}|\{\xi^-\}|\{\mu^+\}|\{\mu^-\}) =& S_n(\{\xi^+\}|\{\mu^+\}|\{\mu^-\}) \prod_{j=1}^{m-n}\Bigg[a(\xi_j^-)\prod_{k=1}^{n} f(\xi_k^+,\xi_j^-)\prod_{k=1}^{M-n} f(\mu_k^-,\xi_j^-) \\ &+ \kappa d(\xi_j^-)\prod_{k=1}^{n}\{-f(\xi_j^-,\xi_k^+)\}\prod_{k=1}^{M-n}\{-f(\xi_j^-,\mu_k^-)\}\Bigg]. \end{aligned} \tag{3.3}$$

Here S_n is defined as

$$S_n(\{\xi^+\}|\{\mu^+\}|\{\mu^-\}) = \frac{\prod_{j,k=1}^n \operatorname{sh}(\xi_j^+ - \mu_k^+ + \eta)}{\prod_{j<k}^n \left[\operatorname{sh}(\mu_k^+ - \mu_j^+)\operatorname{sh}(\xi_j^+ - \xi_k^+)\right]} \det{}_n M_{jk}$$

with

$$\begin{aligned} M_{jk} =& a(\mu_j^+) t(\xi_k^+, \mu_j^+) \prod_{a=1}^{M-n} f(\mu_a^-, \mu_j^+) \\ &- \kappa d(\mu_j^+) t(\mu_j^+, \xi_k^+) \prod_{a=1}^{M-n} \{-f(\mu_j^+, \mu_a^-)\} \prod_{b=1}^{n} \left\{ -\frac{\operatorname{sh}(\mu_j^+ - \xi_b^+ + \eta)}{\operatorname{sh}(\mu_j^+ - \xi_b^+ - \eta)} \right\}. \end{aligned} \tag{3.4}$$

The functions $f(\lambda, \mu)$ and $t(\lambda, \mu)$ appearing in (3.3) *and* (3.4) *are, respectively, given by*

$$f(\lambda, \mu) = \frac{\operatorname{sh}(\lambda - \mu + \eta)}{\operatorname{sh}(\lambda - \mu)}, \quad t(\lambda, \mu) = \frac{\operatorname{sh}\eta}{\operatorname{sh}(\lambda - \mu)\operatorname{sh}(\lambda - \mu + \eta)}.$$

Compared with that for the XXZ chain [9], some sign factors appear in the second term of (3.3) and (3.4), which originate from the fermionic nature of the present system. By setting $\kappa = 1$ and applying the above formula to (3.1), one has

$$\Phi_N(\xi) = -\sum_{n=0}^{m-1} \sum_{\substack{\{\widetilde{\lambda}\}=\{\widetilde{\lambda}^+\}\cup\{\widetilde{\lambda}^-\} \\ \{\widetilde{\xi}\}=\{\widetilde{\xi}^+\}\cup\{\widetilde{\xi}^-\} \\ |\widetilde{\lambda}^+|=|\widetilde{\xi}^+|=n}} \frac{R_n(\{\widetilde{\xi}^+\}|\{\widetilde{\xi}^-\}|\{\widetilde{\lambda}^+\}|\{\widetilde{\lambda}^-\})\langle\{\widetilde{\xi}^+\}\cup\{\widetilde{\lambda}^-\}|B(\xi_{m+1})|\{\lambda\}\rangle}{\langle\{\lambda\}|\{\lambda\}\rangle \prod_{j=1}^{m+1}\Lambda_0(\xi_j)}. \tag{3.5}$$

Here some new notations are adopted:

$$(\widetilde{\lambda}_1, \dots, \widetilde{\lambda}_{N/2+1}) = (\lambda_1, \dots, \lambda_{N/2}, \xi_1), \quad (\widetilde{\xi}_1, \dots, \widetilde{\xi}_{m-1}) = (\xi_2, \dots, \xi_m).$$

Next we evaluate the action of $B(\xi_{m+1})$ on $\langle\{\widetilde{\xi}^+\}\cup\{\widetilde{\lambda}^-\}|$ by using the formula [21]:

$$\begin{aligned} \langle\Omega|\prod_{j=1}^{M} C(\mu_j) B(\mu_{M+1}) =& (-1)^{M-1} \sum_{l=1}^{M+1} d(\mu_l) \frac{\prod_{k=1}^{M}\operatorname{sh}(\mu_l - \mu_k + \eta)}{\prod_{\substack{k=1\\k\neq l}}^{M+1}\operatorname{sh}(\mu_l - \mu_k)} \\ &\times \sum_{\substack{l'=1\\l'\neq l}}^{M+1} \frac{a(\mu_{l'})}{\operatorname{sh}(\mu_{M+1} - \mu_{l'} + \eta)} \frac{\prod_{\substack{j=1\\j\neq l}}^{M+1}\operatorname{sh}(\mu_j - \mu_{l'} + \eta)}{\prod_{\substack{j=1\\j\neq l,l'}}^{M+1}\operatorname{sh}(\mu_j - \mu_{l'})} \langle\Omega| \prod_{\substack{j=1\\j\neq l,l'}}^{M+1} C(\mu_j), \end{aligned} \tag{3.6}$$

where $\{\mu_j\}_{j=1}^{M+1}$ are arbitrary complex numbers. One sees that the resulting equation consists of the ratio of scalar products such as $\langle\{\xi^+\}\cup\{\lambda^-\}|\{\lambda\}\rangle/\langle\{\lambda\}|\{\lambda\}\rangle$, where $\{\lambda\} = \{\lambda_j^+\}_{j=1}^n \cup \{\lambda_j^-\}_{j=1}^{N/2-n}$ and $\{\xi_j^+\}_{j=1}^n \in \{\xi_j\}_{j=1}^{m+1}$ (see the appendix for detail). In fact, this quantity can be calculated by the following determinant representation of the scalar product.

Proposition 3.2 [21]. *The scalar product between a Bethe state and an arbitrary state*

$$\mathbb{S}_M(\{\mu\}|\{\lambda\}) = \langle\Omega|\prod_{j=1}^{M} C(\mu_j)\prod_{j=1}^{M} B(\lambda_j)|\Omega\rangle$$

can be expressed as follows:

$$\mathbb{S}_M(\{\mu\}|\{\lambda\}) = (-1)^{\frac{M(M-1)}{2}} \frac{\prod_{j=1}^{M} d(\lambda_j)a(\mu_j)\prod_{j,k=1}^{M}\mathrm{sh}(\lambda_j-\mu_k+\eta)}{\prod_{j<k}^{M}\mathrm{sh}(\lambda_j-\lambda_k)\,\mathrm{sh}(\mu_k-\mu_j)}\det{}_M\Psi(\{\mu\}|\{\lambda\}),$$

where $\{\lambda_j\}_{j=1}^{M}$ are Bethe roots, $\{\mu_j\}_{j=1}^{M}$ are arbitrary complex parameters. The $M\times M$ matrix $\Psi(\{\mu\}|\{\lambda\})$ is defined by

$$\Psi_{jk}(\{\mu\}|\{\lambda\}) = t(\lambda_j,\mu_k) - (-1)^M t(\mu_k,\lambda_j)\frac{d(\mu_k)}{a(\mu_k)}\prod_{a=1}^{M}\frac{\mathrm{sh}(\mu_k-\lambda_a+\eta)}{\mathrm{sh}(\mu_k-\lambda_a-\eta)},$$

and $\det_M$ *denotes the determinant of an $M\times M$ matrix.*

Applying this, and using the same technique proposed in [15], one obtains the ratio of the scalar products $\langle\{\xi^+\}\cup\{\lambda^-\}|\{\lambda\}\rangle/\langle\{\lambda\}|\{\lambda\}\rangle$:

$$\begin{aligned}\frac{\langle\{\xi^+\}\cup\{\lambda^-\}|\{\lambda\}\rangle}{\langle\{\lambda\}|\{\lambda\}\rangle} =& \prod_{j=1}^{n}\left[\frac{a(\xi_j^+)(1+\mathfrak{a}(\xi_j))}{a(\lambda_j^+)\mathfrak{a}'(\lambda_j^+)}\right]\prod_{j=1}^{\frac{N}{2}-n}\prod_{k=1}^{n}\left[\frac{f(\lambda_j^-,\xi_k^+)}{f(\lambda_j^-,\lambda_k^+)}\right]\\ &\times\prod_{j,k=1}^{n}\left[\frac{\mathrm{sh}(\lambda_j^+-\xi_k^++\eta)}{\mathrm{sh}(\lambda_j^+-\lambda_k^++\eta)}\right]\prod_{j<k}^{n}\left[\frac{\mathrm{sh}(\lambda_j^+-\lambda_k^+)}{\mathrm{sh}(\xi_j^+-\xi_k^+)}\right]\det{}_n G(\lambda_j^+,\xi_k^+),\end{aligned}\quad (3.7)$$

where the function $G(\lambda,\xi)$ satisfies the following linear integral equation

$$G(\lambda,\xi) = t(\xi,\lambda) + \int_{\mathcal{C}}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{\mathrm{sh}(2\eta)}{\mathrm{sh}(\lambda-\omega+\eta)\,\mathrm{sh}(\lambda-\omega-\eta)}\frac{G(\omega,\xi)}{1+\mathfrak{a}(\omega)},\quad (3.8)$$

which can also be written in terms of $\overline{\mathfrak{a}}(\lambda)$ as

$$G(\lambda,\xi) = -t(\lambda,\xi) - \int_{\mathcal{C}}\frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\frac{\mathrm{sh}(2\eta)}{\mathrm{sh}(\lambda-\omega+\eta)\,\mathrm{sh}(\lambda-\omega-\eta)}\frac{G(\omega,\xi)}{1+\overline{\mathfrak{a}}(\omega)}.\quad (3.9)$$

Applying all the steps described above, we find that (3.1) can be reduced to sums over the partitions of the sets $\{\lambda\}$ and $\{\xi\}$, and its summand consists of determinants of matrices constructed by functions of $\{\lambda\}$ and $\{\xi\}$ (see (A.3) for example). In fact, by using the technique as in [15], these sums can be transformed to multiple integrals on the canonical contour $\mathcal{C}$, where the Trotter limit can be taken analytically. The derivation is straightforward but has a lot of steps, here we only write down the final result. Namely, the function $\Phi_N(\{\xi\})$

(3.1) is represented by the following multiple integral:

$$\Phi_N(\{\xi\}) = \sum_{n=0}^{m-1} \frac{(-1)^m}{n!(n+1)!} \int_{\Gamma^{n+1}} \prod_{j=1}^{n+1} \frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}} \frac{\mathrm{sh}(\zeta_j - \xi_1 - \eta)}{\mathfrak{b}_-(\zeta_j)\,\mathrm{sh}(\zeta_j - \xi_{m+1})}$$
$$\times \int_{\mathcal{C}^n} \prod_{j=1}^{n} \frac{\mathrm{d}\omega_j}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))} \frac{\mathfrak{b}_-(\omega_j)\,\mathrm{sh}(\omega_j - \xi_{m+1})}{\mathrm{sh}(\omega_j - \xi_1 - \eta)} \int_{\mathcal{C}} \frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))} \int_{\mathcal{C}} \frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}$$
$$\times \frac{\prod_{j=1}^{n+1}[\mathrm{sh}(\omega_{n+1} - \zeta_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \zeta_j - \eta)]}{\prod_{j=1}^{n}[\mathrm{sh}(\omega_{n+1} - \omega_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \omega_j - \eta)]} \frac{W_n^-(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1} - \omega_{n+2} + \eta)}$$
$$\times \det_{n+1} M_{jk}^-(\{\omega\}|\{\zeta\}) \det_{n+2}\left[G(\omega_j,\zeta_1),\ldots,G(\omega_j,\zeta_{n+1}),G(\omega_j,\xi_{m+1})\right], \tag{3.10}$$

where

$$\mathfrak{b}_\pm(\omega) = \prod_{j=2}^{m} \frac{\mathrm{sh}(\omega - \xi_j)}{\mathrm{sh}(\omega - \xi_j \pm \eta)},$$
$$W_n^\pm(\{\omega\}|\{\zeta\}) = \frac{\prod_{j=1}^{n}\prod_{k=1}^{n+1} \mathrm{sh}(\omega_j - \zeta_k \pm \eta)\,\mathrm{sh}(\zeta_k - \omega_j \pm \eta)}{\prod_{j=1}^{n}\prod_{k=1}^{n} \mathrm{sh}(\omega_j - \omega_k \pm \eta) \prod_{j=1}^{n+1}\prod_{k=1}^{n+1} \mathrm{sh}(\zeta_j - \zeta_k \pm \eta)}, \tag{3.11}$$

and $M^-(\{\omega\}|\{\zeta\})$ is an $(n+1)\times(n+1)$ matrix whose matrix elements are given by

$$M_{jk}^- = \begin{cases} t(\omega_j,\zeta_k) + t(\zeta_k,\omega_j) \prod_{a=1}^{n} \frac{\mathrm{sh}(\omega_a - \omega_j - \eta)}{\mathrm{sh}(\omega_j - \omega_a - \eta)} \prod_{b=1}^{n+1} \frac{\mathrm{sh}(\omega_j - \zeta_b - \eta)}{\mathrm{sh}(\zeta_b - \omega_j - \eta)} & \text{for } j \le n \\ t(\xi_1,\zeta_k) & \text{for } j = n+1 \end{cases}.$$

$\mathfrak{a}(\lambda)(=1/\overline{\mathfrak{a}}(\lambda))$ and $G(\lambda,\zeta)$ satisfy the integral equations (2.14) and (3.8), respectively. $\mathcal{C}$ is the canonical contour defined as in figure 1, and Γ encircles ξ and does not contain any other singularities.

The one-particle Green's function can be obtained from the above expression by taking the limits $\{\xi\} \to 0$ and $N \to \infty$ (see (3.2)). The latter means to take $\mathfrak{a}(\lambda)$ as a function satisfying (2.15). We thus arrive at

Theorem 3.1 *The equal-time one-particle Green's function of the spinless fermion model at finite temperature has the following multiple integral representation,*

$$\langle c_1 c_{m+1}^\dagger \rangle = \sum_{n=0}^{m-1} \frac{(-1)^m}{n!(n+1)!} \int_{\Gamma^{n+1}} \prod_{j=1}^{n+1} \frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}} \left(\frac{\mathrm{sh}(\zeta_j - \eta)}{\mathrm{sh}(\zeta_j)}\right)^m$$
$$\times \int_{\mathcal{C}^n} \prod_{j=1}^{n} \frac{\mathrm{d}\omega_j}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))} \left(\frac{\mathrm{sh}(\omega_j)}{\mathrm{sh}(\omega_j - \eta)}\right)^m \int_{\mathcal{C}} \frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))} \int_{\mathcal{C}} \frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}$$
$$\times \frac{\prod_{j=1}^{n+1}[\mathrm{sh}(\omega_{n+1} - \zeta_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \zeta_j - \eta)]}{\prod_{j=1}^{n}[\mathrm{sh}(\omega_{n+1} - \omega_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \omega_j - \eta)]} \frac{W_n^-(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1} - \omega_{n+2} + \eta)}$$
$$\times \det_{n+1} M_{jk}^-(\{\omega\}|\{\zeta\})\Big|_{\xi_1 \to 0} \det_{n+2}\left[G(\omega_j,\zeta_1),\ldots,G(\omega_j,\zeta_{n+1}),G(\omega_j,0)\right], \tag{3.12}$$

where $\mathfrak{a}(\lambda) = 1/(\overline{\mathfrak{a}}(\lambda))$ *and* $G(\lambda,\zeta)$ *satisfy the integral equation* (2.15) *and* (3.8)*, respectively.* $\mathcal{C}$ *is the canonical contour and* Γ *surrounds the point* 0.

Using the identity

$$\frac{1}{1+\mathfrak{a}(\omega)} = 1 - \frac{1}{1+\overline{\mathfrak{a}}(\omega)}, \tag{3.13}$$

we can convert the above multiple integral representation into another form. Namely, inserting the decomposition (3.13) into the part $\prod_{j=1}^{n} 1/(1+\mathfrak{a}(\omega_j))$ of (3.10), and then performing the integrals over ζ_j, we transform them to sums over the partition of the set $\{\xi\}$. Resumming the results in a similar way as in the appendix, we have

$$\begin{aligned}
\Phi_N(\{\xi\}) = &\sum_{n=0}^{m} \frac{(-1)^n}{n!(n+1)!} \int_{\Gamma^{n+1}} \prod_{j=1}^{n+1} \frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}} \frac{\mathrm{sh}(\zeta_j - \xi_1 + \eta)}{\mathfrak{b}_+(\zeta_j)\,\mathrm{sh}(\zeta_j - \xi_{m+1})} \\
&\times \int_{\mathcal{C}^n} \prod_{j=1}^{n} \frac{\mathrm{d}\omega_j}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_j))} \frac{\mathfrak{b}_+(\omega_j)\,\mathrm{sh}(\omega_j - \xi_{m+1})}{\mathrm{sh}(\omega_j - \xi_1 + \eta)} \int_{\mathcal{C}} \frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))} \int_{\mathcal{C}} \frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))} \\
&\times \frac{\prod_{j=1}^{n+1} [\mathrm{sh}(\omega_{n+1} - \zeta_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \zeta_j - \eta)]}{\prod_{j=1}^{n} [\mathrm{sh}(\omega_{n+1} - \omega_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \omega_j - \eta)]} \frac{W_n^+(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1} - \omega_{n+2} + \eta)} \\
&\times \det{}_{n+1} M_{jk}^+(\{\omega\}|\{\zeta\}) \det{}_{n+2} \left[G(\omega_j, \zeta_1), \ldots, G(\omega_j, \zeta_{n+1}), G(\omega_j, \xi_{m+1})\right],
\end{aligned}$$

where $\mathfrak{b}_+(\omega)$ and $W_n^+(\{\omega\}|\{\zeta\})$ are defined in (3.11). $M^+(\{\omega\}|\{\zeta\})$ is an $(n+1)\times(n+1)$ matrix whose matrix elements are given by

$$M_{jk}^+ = \begin{cases} t(\zeta_k, \omega_j) + t(\omega_j, \zeta_k) \prod_{a=1}^{n} \frac{\mathrm{sh}(\omega_a - \omega_j + \eta)}{\mathrm{sh}(\omega_j - \omega_a + \eta)} \prod_{b=1}^{n+1} \frac{\mathrm{sh}(\omega_j - \zeta_b + \eta)}{\mathrm{sh}(\zeta_b - \omega_j + \eta)} & \text{for } j \le n \\ t(\zeta_k, \xi_1) & \text{for } j = n+1 \end{cases}.$$

Taking the homogeneous and the Trotter limits, we have another multiple integral representing the one-particle Green's function.

Corollary 3.1 *The equal-time one-particle Green's function of the spinless fermion model at finite temperature has another multiple integral representation:*

$$\begin{aligned}
\langle c_1 c_{m+1}^\dagger \rangle = &\sum_{n=0}^{m-1} \frac{(-1)^n}{n!(n+1)!} \int_{\Gamma^{n+1}} \prod_{j=1}^{n+1} \frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}} \left(\frac{\mathrm{sh}(\zeta_j + \eta)}{\mathrm{sh}(\zeta_j)}\right)^m \\
&\times \int_{\mathcal{C}^n} \prod_{j=1}^{n} \frac{\mathrm{d}\omega_j}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_j))} \left(\frac{\mathrm{sh}(\omega_j)}{\mathrm{sh}(\omega_j + \eta)}\right)^m \int_{\mathcal{C}} \frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))} \int_{\mathcal{C}} \frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))} \\
&\times \frac{\prod_{j=1}^{n+1} [\mathrm{sh}(\omega_{n+1} - \zeta_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \zeta_j - \eta)]}{\prod_{j=1}^{n} [\mathrm{sh}(\omega_{n+1} - \omega_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \omega_j - \eta)]} \frac{W_n^+(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1} - \omega_{n+2} + \eta)} \\
&\times \det{}_{n+1} M_{jk}^+(\{\omega\}|\{\zeta\})\Big|_{\xi_1 \to 0} \det{}_{n+2} \left[G(\omega_j, \zeta_1), \ldots, G(\omega_j, \zeta_{n+1}), G(\omega_j, 0)\right],
\end{aligned} \tag{3.14}$$

where $\overline{\mathfrak{a}}(\lambda) = 1/\mathfrak{a}(\lambda)$ *and* $G(\lambda, \zeta)$ *satisfy the integral equations* (2.16) *and* (3.9)*, respectively.*

4 Special cases

In this section, we evaluate the three special cases of the multiple integral representation: the zero-temperature and the infinite-temperature and the free fermion limits.

4.1 Zero-temperature limit

First let us consider the zero-temperature limit. Here we restrict ourselves on the off-critical case $\Delta > 0$, and set $\eta < 0$ as in [21]. The critical case $0 \le \Delta \le 1$, of course, can be treated by just changing the definition of the integration contour $\mathcal{C}$ as in figure 1.

Shifting the variables in (3.14) by $\omega_j \to \omega_j - \eta/2$ and $\zeta_j \to \zeta_j - \eta/2$, we deal with the integrals on the contour $\Gamma_{\eta/2}$ and $\mathcal{C}_0 \cup \mathcal{C}_\eta$, where $\Gamma_{\eta/2}$ encircles the point $\eta/2$; $\mathcal{C}_0$ and $\mathcal{C}_{\eta/2}$ are defined as $\mathcal{C}_0 = [-\pi\mathrm{i}/2, \pi\mathrm{i}/2]$ and $\mathcal{C}_\eta = [\eta + \pi\mathrm{i}/2, \eta - \pi\mathrm{i}/2]$, respectively. A close inspection of the auxiliary functions $\mathfrak{a}(\lambda)$ and $\overline{\mathfrak{a}}(\lambda)$ for $\mu_{\mathrm{c}} > 0$ and $\eta < 0$ at the zero-temperature limit $T \to 0$ leads to

$$\frac{1}{1+\mathfrak{a}(\lambda-\frac{\eta}{2})} \overset{T\to 0}{\longrightarrow} \begin{cases} 1 & \text{for } \lambda \in \overline{\mathcal{L}} \\ 0 & \text{for } \lambda \in \mathcal{L} \end{cases}, \quad \frac{1}{1+\overline{\mathfrak{a}}(\lambda-\frac{\eta}{2})} \overset{T\to 0}{\longrightarrow} \begin{cases} 0 & \text{for } \lambda \in \overline{\mathcal{L}} \\ 1 & \text{for } \lambda \in \mathcal{L} \end{cases}, \tag{4.1}$$

where $\mathcal{L} = [-q_{\mu_{\mathrm{c}}}, q_{\mu_{\mathrm{c}}}]$ and $\overline{\mathcal{L}} = (\mathcal{C}_0 \cup \mathcal{C}_\eta) \setminus \mathcal{L}$. Note that the Fermi point $q_{\mu_{\mathrm{c}}}$ is an imaginary number ($\operatorname{Im} q_{\mu_{\mathrm{c}}} > 0$) depending on the chemical potential μ_{c}. Substituting this into (3.9) and shifting the variables as above, one has

$$G\left(\lambda - \frac{\eta}{2}, \zeta - \frac{\eta}{2}\right) = -t(\lambda,\zeta) + \int_{-\mathcal{L}} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{\mathrm{sh}(2\eta) G(\omega - \frac{\eta}{2}, \zeta - \frac{\eta}{2})}{\mathrm{sh}(\lambda-\omega+\eta)\,\mathrm{sh}(\lambda-\omega-\eta)}.$$

Comparing this with equation (2.24) in [21], one can identify $G(\lambda,\zeta)$ with the density function $\rho(\lambda,\zeta)$:

$$G\left(\lambda - \frac{\eta}{2}, \zeta - \frac{\eta}{2}\right) = 2\pi\mathrm{i}\rho(\lambda,\zeta). \tag{4.2}$$

Inserting both (4.2) and (4.1) into (3.14), we finally obtain

$$\begin{aligned}
\lim_{T\to 0} \langle c_1 c^\dagger_{m+1} \rangle = &-\sum_{n=0}^{m-1} \frac{1}{n!(n+1)!} \int_{\Gamma_{\frac{\eta}{2}}} \prod_{j=1}^{n+1} \frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}} \int_{-\mathcal{L}} \mathrm{d}^{n+1}\omega \int_{\overline{\mathcal{L}}} \mathrm{d}\omega_{n+2} \prod_{j=1}^{n+1} \left(\frac{\mathrm{sh}(\zeta_j + \frac{\eta}{2})}{\mathrm{sh}(\zeta_j - \frac{\eta}{2})} \right)^m \\
&\times \prod_{j=1}^{n} \left(\frac{\mathrm{sh}(\omega_j - \frac{\eta}{2})}{\mathrm{sh}(\omega_j + \frac{\eta}{2})} \right)^m \frac{\prod_{j=1}^{n+1} [\mathrm{sh}(\omega_{n+1} - \zeta_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \zeta_j - \eta)]}{\prod_{j=1}^{n} [\mathrm{sh}(\omega_{n+1} - \omega_j + \eta)\,\mathrm{sh}(\omega_{n+2} - \omega_j - \eta)]} \frac{W_n^+(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1} - \omega_{n+2} + \eta)} \\
&\times \det{}_{n+1} M^+_{jk}(\{\omega\}|\{\zeta\}) \Big|_{\xi_1 \to \frac{\eta}{2}} \det{}_{n+2} \left[\rho(\omega_j, \zeta_1), \ldots, \rho(\omega_j, \zeta_{n+1}), \rho(\omega_j, \eta/2) \right].
\end{aligned}$$

The above representation completely agrees with equation (4.18) in [21].

4.2 Infinite-temperature limit

Next we would like to deal with the infinite-temperature case $T = \infty$, where the function $\Phi_N(\{\xi\})$ (3.10) does not depend on the Trotter number N (note that $\mathfrak{a}(\lambda) = 1$ and $\overline{\mathfrak{a}}(\lambda) = 1$). Therefore the integrals can be explicitly evaluated by just applying the residue theorem to the poles of integrand. The result reads

$$\lim_{T\to\infty} \langle c_1 c^\dagger_{m+1} \rangle = 0,$$

as one expected.

4.3 Free fermion point

Finally we investigate the representation (3.12) at the free fermion point $\Delta = 0$. Set $\eta = \pi\mathrm{i}/2$. Then the integral kernel in (2.15) and (3.8) becomes zero. Hence the two functions $\mathfrak{a}(\omega)(= 1/\overline{\mathfrak{a}}(\omega))$ and $G(\omega,\zeta)$ can be explicitly written as

$$\mathfrak{a}(\omega) = \exp\left\{-\frac{1}{T}\left(\mu_{\mathrm{c}} + \frac{2\mathrm{i}t}{\mathrm{sh}(2\omega)}\right)\right\}, \quad G(\omega,\zeta) = -\frac{2}{\mathrm{sh}(2(\omega-\zeta))}.$$

Since $M_{jk} = 0$ for $j \le n$, one observes all the terms $n \ge 1$ vanish. Applying the decomposition $1/(1+\overline{\mathfrak{a}}(\omega_1)) = 1 - 1/(1+\mathfrak{a}(\omega_1))$, one finds that the integral including $1/(1+\mathfrak{a}(\omega_1))$ is equal to zero since the integrand is antisymmetric with respect to ω_1 and ω_2. After shifting the variables $\zeta_j \to \zeta_j + \pi\mathrm{i}/4$ and $\omega_j \to \omega_j + \pi\mathrm{i}/4$, one obtains

$$\begin{aligned}\langle c_1 c^{\dagger}_{m+1}\rangle = -8(-1)^m \int_{\Gamma_{-\frac{\pi}{4}\mathrm{i}}} \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} \left[\frac{\mathrm{sh}(\zeta-\frac{\pi}{4}\mathrm{i})}{\mathrm{sh}(\zeta+\frac{\pi}{4}\mathrm{i})}\right]^m \frac{1}{\mathrm{sh}(2(\zeta+\frac{\pi}{4}\mathrm{i}))} \int_{\mathcal{C}'} \frac{\mathrm{d}\omega_1}{2\pi\mathrm{i}} \int_{\mathcal{C}'} \frac{\mathrm{d}\omega_2}{2\pi\mathrm{i}} \frac{1}{1+\mathfrak{a}(\omega_2+\frac{\pi}{4}\mathrm{i})} \\ \times \frac{\mathrm{ch}(\omega_1-\zeta)\,\mathrm{ch}(\omega_2-\zeta)}{\mathrm{ch}(\omega_1-\omega_2)} \left[\frac{1}{\mathrm{sh}(2(\omega_1-\zeta))\,\mathrm{sh}(2(\omega_2+\frac{\pi}{4}\mathrm{i}))} - \frac{1}{\mathrm{sh}(2(\omega_2-\zeta))\,\mathrm{sh}(2(\omega_1+\frac{\pi}{4}\mathrm{i}))}\right],\end{aligned}$$

where $\Gamma_{-\pi\mathrm{i}/4}$ surrounds the point $\zeta = -\pi\mathrm{i}/4$; $\mathcal{C}' = -\mathcal{C}_0 \cup \mathcal{C}_{-\pi\mathrm{i}/2}$; $\mathcal{C}_0 = [-\infty,\infty]$; $\mathcal{C}_{-\pi\mathrm{i}/2} = [-\pi\mathrm{i}/2-\infty, -\pi\mathrm{i}/2+\infty]$. The integral with respect to ω_1 can be easily evaluated via the residue theorem applied to the poles at $\omega = -\pi\mathrm{i}/4$ and ζ. Then taking into account the pole outside the contour $\Gamma_{-\pi\mathrm{i}/4}$ i.e. at the point $\zeta = \omega_2$, we compute the integral with respect to ζ. It reads

$$\begin{aligned}\langle c_1 c^{\dagger}_{m+1}\rangle =& 2(-1)^m \int_{\mathcal{C}'} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{1}{1+\mathfrak{a}(\omega+\frac{\pi}{4}\mathrm{i})} \left[\frac{\mathrm{sh}(\omega-\frac{\pi}{4}\mathrm{i})}{\mathrm{sh}(\omega+\frac{\pi}{4}\mathrm{i})}\right]^m \frac{1}{\mathrm{sh}(2(\omega+\frac{\pi}{4}\mathrm{i}))} \\ =& -2(-1)^m \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{1}{1+\mathfrak{a}(\omega+\frac{\pi}{4}\mathrm{i})} \left[\frac{\mathrm{sh}(\omega-\frac{\pi}{4}\mathrm{i})}{\mathrm{sh}(\omega+\frac{\pi}{4}\mathrm{i})}\right]^m \frac{1}{\mathrm{sh}(2(\omega+\frac{\pi}{4}\mathrm{i}))} \\ &+ 2(-1)^m \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}} \frac{1}{1+\mathfrak{a}(\omega-\frac{\pi}{4}\mathrm{i})} \left[-\frac{\mathrm{sh}(\omega+\frac{\pi}{4}\mathrm{i})}{\mathrm{sh}(\omega-\frac{\pi}{4}\mathrm{i})}\right]^m \frac{1}{\mathrm{sh}(2(\omega-\frac{\pi}{4}\mathrm{i}))}.\end{aligned}$$

Changing the variable $\cosh(2\omega) = 1/\cos p$ ($p \in [-\pi/2,\pi/2]$) for the first term in the second equality, and $\cosh(2\omega) = -1/\cos p$ ($p \in [-\pi,-\pi/2] \cup [\pi/2,\pi]$) for the second term, we obtain

$$\langle c_1 c^{\dagger}_{m+1}\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}p\mathrm{e}^{\mathrm{i}mp}}{1+\exp\left[-\frac{\mu_{\mathrm{c}}}{T} - \frac{2t}{T}\cos p\right]} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{d}p\mathrm{e}^{\mathrm{i}mp}}{1+\exp\left[\frac{\mu_{\mathrm{c}}}{T} + \frac{2t}{T}\cos p\right]}. \tag{4.3}$$

We note that the above expression reproduces the well-known result (see [28] for example). Of course, (4.3) can also be derived by starting from (3.14).

5 Discussion

We have derived the multiple integral representation for the equal-time one-particle Green's function of the spinless fermion model at finite temperature. Unfortunately, the explicit evaluation of the multiple integrals still remains a difficult task, except for some special cases

considered here. Nevertheless, we believe that the method provided in this paper should be useful for the future study of the correlation functions of the fermionic systems.

For instance, from (2.8) one sees that the long-distance behavior of the two-point correlation functions can be calculated by taking the ratio between the largest and the subleading eigenvalues of the quantum transfer matrix. For the one-particle Green's function, it reads (up to the sign)

$$\langle c_1 c_{m+1}^\dagger \rangle \sim 2A_0 \cos(k_{\rm F}(m-1)) \exp\left[-\frac{m-1}{\xi}\right],$$

with

$$k_{\rm F} = {\rm Im}\left[\ln\frac{\Lambda_1(0)}{\Lambda_0(0)}\right], \quad -\frac{1}{\xi} = {\rm Re}\left[\ln\frac{\Lambda_1(0)}{\Lambda_0(0)}\right], \quad A_0 = \left|\frac{\langle\Psi_0|B(0)|\Psi_1\rangle}{\Lambda_0(0)}\right|^2,$$

where $\Lambda_1(0)$ is the leading eigenvalue for the sector $M = N/2-1$ (see (2.11) and (2.12)) and $|\Psi_1\rangle$ is the corresponding (normalized) eigenvector. In fact the finite temperature correlation length ξ has already been calculated in [22]. The evaluation of the amplitude A_0 by using (3.6) and Proposition 3.2 is quite important problem.

It is also interesting to extend our result to the time dependent case. This is possible by combining the present method with the solution of the quantum inverse scattering problem for the operator c_j. It is evidently worth while to extract the long-distance and long-time behavior of the correlation functions at any finite temperatures.

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Appendix. Derivation of multiple integral (3.10)

We describe how the multiple integral representation (3.10) is derived. Applying the relation (3.6) to the term $\langle\Omega|\prod_{j=1}^{n} C(\widetilde{\xi}_j^+)\prod_{k=1}^{N/2+1-n} C(\widetilde{\lambda}_k^-)B(\xi_{m+1})$ in the r.h.s of (3.5), we split $\Phi_N(\{\xi\})$ into four parts according to whether the arguments of the functions $a(x_a)$ and $d(x_d)$ appearing in the resulting equation are Bethe roots $\{\lambda\}$ or inhomogeneous parameters $\{\xi\}$:

$$\begin{aligned}\Phi_N(\{\xi\}) =& F_1(\{x_a\}_{\in\{\lambda\}}|\{x_d\}_{\in\{\lambda\}}) + F_2(\{x_a\}_{\in\{\lambda\}}|\{x_d\}_{\in\{\xi\}}) \\ &+ F_3(\{x_a\}_{\in\{\xi\}}|\{x_d\}_{\in\{\lambda\}}) + F_4(\{x_a\}_{\in\{\xi\}}|\{x_d\}_{\in\{\xi\}}).\end{aligned} \tag{A.1}$$

First we consider the function F_1 which can further be divided into two parts according to whether $\xi_1 \in \{\widetilde{\lambda}^+\}$ or $\xi_1 \in \{\widetilde{\lambda}^-\}$: $F_1 = F_{\xi_1\in\{\widetilde{\lambda}^+\}} + F_{\xi_1\in\{\widetilde{\lambda}^-\}}$, where

$$F_{\xi_1\in\{\widetilde{\lambda}^+\}} = (-1)^{\frac{N}{2}+1} \sum_{n=1}^{m-1} \sum_{\substack{\{\widetilde{\lambda}\}=\{\widetilde{\lambda}^+\}\cup\{\lambda^-\}\\ \{\widetilde{\xi}\}=\{\widetilde{\xi}^+\}\cup\{\widetilde{\xi}^-\}\\ |\widetilde{\lambda}^+|=|\widetilde{\xi}^+|=n}} \sum_{l=1}^{\frac{N}{2}-n+1} \sum_{\substack{l'=1\\ l'\neq l}}^{\frac{N}{2}-n+1} \frac{H_n^{(1)}(\{\lambda^-\}|\{\widetilde{\xi}^+\}) H_n^{(2)}(\{\lambda^-\}|\{\widetilde{\xi}^+\})}{\langle\{\lambda\}|\{\lambda\}\rangle \prod_{j=1}^{m+1}\Lambda_0(\xi_j)}$$

$$\times R_n(\{\widetilde{\xi}^+\}|\{\widetilde{\xi}^-\}|\{\widetilde{\lambda}^+\}|\{\lambda^-\})\langle\Omega| \prod_{\substack{j=1\\ j\neq l,l'}}^{\frac{N}{2}-n+1} C(\lambda_j^-) \prod_{j=1}^{n} C(\widetilde{\xi}_j^+) C(\xi_{m+1}) \prod_{j=1}^{\frac{N}{2}} B(\lambda_j)|\Omega\rangle \tag{A.2}$$

with

$$H_n^{(1)}(\{\lambda^-\}|\{\widetilde{\xi}^+\}) = \frac{d(\lambda_l^-)\,\mathrm{sh}(\eta) \prod_{\substack{j=1\\ j\neq l}}^{\frac{N}{2}-n+1} f(\lambda_l^-,\lambda_j^-) \prod_{j=1}^{n} f(\lambda_l^-,\widetilde{\xi}_j^+)}{\mathrm{sh}(\lambda_l^- - \xi_{m+1})},$$

$$H_n^{(2)}(\{\lambda^-\}|\{\widetilde{\xi}^+\}) = \frac{a(\lambda_{l'}^-)\,\mathrm{sh}(\eta) f(\xi_{m+1},\lambda_{l'}^-) \prod_{\substack{j=1\\ j\neq l,l'}}^{\frac{N}{2}-n+1} f(\lambda_j^-,\lambda_{l'}^-) \prod_{j=1}^{n} f(\widetilde{\xi}_j^+,\lambda_{l'}^-)}{\mathrm{sh}(\xi_{m+1} - \lambda_{l'}^- + \eta)},$$

while $F_{\xi_1\in\{\widetilde{\lambda}^-\}}$ is

$$F_{\xi_1\in\{\widetilde{\lambda}^-\}} = (-1)^{\frac{N}{2}+1} \sum_{n=0}^{m-1} \sum_{\substack{\{\widetilde{\lambda}\}=\{\lambda^+\}\cup\{\widetilde{\lambda}^-\}\\ \{\widetilde{\xi}\}=\{\widetilde{\xi}^+\}\cup\{\widetilde{\xi}^-\}\\ |\lambda^+|=|\widetilde{\xi}^+|=n}} \sum_{l=1}^{\frac{N}{2}-n} \sum_{\substack{l'=1\\ l'\neq l}}^{\frac{N}{2}-n} \frac{H_n^{(1)}(\{\widetilde{\lambda}^-\}|\{\widetilde{\xi}^+\}) H_n^{(2)}(\{\widetilde{\lambda}^-\}|\{\widetilde{\xi}^+\})}{\langle\{\lambda\}|\{\lambda\}\rangle \prod_{j=1}^{m+1}\Lambda_0(\xi_j)}$$

$$\times R_n(\{\widetilde{\xi}^+\}|\{\widetilde{\xi}^-\}|\{\lambda^+\}|\{\widetilde{\lambda}^-\})\langle\Omega| \prod_{\substack{j=1\\ j\neq l,l'}}^{\frac{N}{2}-n+1} C(\widetilde{\lambda}_j^-) \prod_{j=1}^{n} C(\widetilde{\xi}_j^+) C(\xi_{m+1}) \prod_{j=1}^{\frac{N}{2}} B(\lambda_j)|\Omega\rangle.$$

Here $\{\lambda^\pm\}$ denote $\{\lambda^\pm\} = \{\widetilde{\lambda}^\pm\}^{\backslash \xi_1}$.

Inserting the relations (2.12), (2.13), (3.3), (3.7) and $\mathfrak{a}(\lambda_j^+) = -1$ into (A.2) and shifting the variable $n \to n+1$, we have

$$F_{\xi_1\in\{\widetilde{\lambda}^+\}} = \sum_{n=0}^{m-2} \sum_{\substack{\{\lambda\}=\{\lambda^+\}\cup\{\lambda^-\}\\ \{\widetilde{\xi}\}=\{\widetilde{\xi}^+\}\cup\{\widetilde{\xi}^-\}\\ |\lambda^+|=|\widetilde{\xi}^+|-1=n}} \sum_{l=1}^{\frac{N}{2}-n} \sum_{\substack{l'=1\\ l'\neq l}}^{\frac{N}{2}-n} \frac{(-1)^n \mathfrak{a}(\lambda_l^-)}{\mathfrak{a}'(\lambda_l^-)\mathfrak{a}'(\lambda_{l'}^-)\prod_{j=1}^{n}\mathfrak{a}'(\lambda_j^+)}$$

$$\times \frac{\widetilde{Y}_n(\{\lambda^+\}|\{\widetilde{\xi}^+\}) Z_n(\{\lambda^+\}|\{\widetilde{\xi}\}) V_n^+(\{\lambda^+\}|\{\widetilde{\xi}^+\}) X_n(\{\lambda^+\}|\{\widetilde{\xi}^+\})}{\mathrm{sh}(\lambda_l^- - \lambda_{l'}^- + \eta)(1+\mathfrak{a}(\xi_1)) \prod_{j=1}^{m-n-2}(1+\mathfrak{a}(\widetilde{\xi}_j^-))}, \tag{A.3}$$

where the functions $\widetilde{Y}_n(\{\lambda^+\}|\{\widetilde{\xi}^+\})$, $V_n^{\pm}(\{\lambda^+\}|\{\widetilde{\xi}^+\})$, $X_n(\{\lambda^+\}|\{\widetilde{\xi}^+\})$ and $Z_n(\{\lambda^+\}|\{\widetilde{\xi}\})$ are defined as follows:

$$\widetilde{Y}_n(\{\lambda^+\}|\{\widetilde{\xi}^+\}) = \frac{\prod_{j=1}^{n}\mathfrak{b}_+(\lambda_j^+)}{\prod_{j=1}^{n+1}\mathfrak{b}'_+(\widetilde{\xi}_j^+)}\frac{\prod_{j=1}^{n}\prod_{k=1}^{n+1}[\operatorname{sh}(\lambda_j^+-\widetilde{\xi}_k^+-\eta)\operatorname{sh}(\lambda_j^+-\widetilde{\xi}_k^++\eta)]}{\prod_{j,k=1}^{n+1}\operatorname{sh}(\widetilde{\xi}_j^+-\widetilde{\xi}_k^++\eta)\prod_{j,k=1}^{n}\operatorname{sh}(\lambda_j^+-\lambda_k^+-\eta)}$$
$$\times\operatorname{det}_{n+1}\widetilde{M}_{jk}\operatorname{det}_{n+2}G(\widehat{\lambda}_j,\widehat{\xi}_k)$$

with

$$\mathfrak{b}_{\pm}(\lambda)=\prod_{j=1}^{m-1}\frac{\operatorname{sh}(\lambda-\widetilde{\xi}_j)}{\operatorname{sh}(\lambda-\widetilde{\xi}_j\pm\eta)}=\prod_{j=2}^{m}\frac{\operatorname{sh}(\lambda-\xi_j)}{\operatorname{sh}(\lambda-\xi_j\pm\eta)},$$
$$\widetilde{M}_{jk}=\begin{cases}t(\widetilde{\xi}_k^+,\lambda_j^+)-t(\lambda_j^+,\widetilde{\xi}_k^+)\prod_{a=1}^{n}\frac{f(\lambda_a^+,\lambda_j^+)}{f(\lambda_j^+,\lambda_a^+)}\prod_{b=1}^{n+1}\frac{f(\lambda_j^+,\widetilde{\xi}_b^+)}{f(\widetilde{\xi}_b^+,\lambda_j^+)} & \text{for } j\le n\\ t(\widetilde{\xi}_k^+,\xi_1)+\mathfrak{a}(\xi_1)t(\xi_1,\widetilde{\xi}_k^+)\prod_{a=1}^{n}\frac{f(\lambda_a^+,\xi_1)}{f(\xi_1,\lambda_a^+)}\prod_{b=1}^{n+1}\frac{f(\xi_1,\widetilde{\xi}_b^+)}{f(\widetilde{\xi}_b^+,\xi_1)} & \text{for } j=n+1\end{cases},$$

and $G(\widehat{\lambda},\widehat{\xi})$ is the solution of the linear integral equation (3.8), where the variables $\{\widehat{\lambda}_j\}_{j=1}^{n+2}$ and $\{\widehat{\xi}_j\}_{j=1}^{n+2}$ are, respectively, assigned as

$$(\widehat{\lambda}_1,\ldots,\widehat{\lambda}_{n+2})=(\lambda_1^+,\ldots,\lambda_n^+,\lambda_l^-,\lambda_{l'}^-),\quad(\widehat{\xi}_1,\ldots,\widehat{\xi}_{n+2})=(\widetilde{\xi}_1^+,\ldots,\widetilde{\xi}_{n+1}^+,\xi_{m+1});$$

$$V_n^{\pm}(\{\lambda^+\}|\{\widetilde{\xi}^+\})=\frac{\prod_{j=1}^{n}\operatorname{sh}(\lambda_j^+-\xi_{m+1})\prod_{j=1}^{n+1}\operatorname{sh}(\widetilde{\xi}_j^+-\xi_1\pm\eta)}{\prod_{j=1}^{n+1}\operatorname{sh}(\widetilde{\xi}_j^+-\xi_{m+1})\prod_{j=1}^{n}\operatorname{sh}(\lambda_j^+-\xi_1\pm\eta)},$$
$$X_n(\{\lambda^+\}|\{\widetilde{\xi}^+\})=\frac{\prod_{j=1}^{n+1}[\operatorname{sh}(\lambda_l^--\widetilde{\xi}_j^++\eta)\operatorname{sh}(\lambda_{l'}^--\widetilde{\xi}_j^+-\eta)]}{\prod_{j=1}^{n}[\operatorname{sh}(\lambda_l^--\lambda_j^++\eta)\operatorname{sh}(\lambda_{l'}^--\lambda_j^+-\eta)]},$$
$$Z_n(\{\lambda^+\}|\{\widetilde{\xi}\})=\prod_{j=1}^{|\widetilde{\xi}^-|}\left[1-\mathfrak{a}(\widetilde{\xi}_j^-)\prod_{k=1}^{n}\frac{f(\lambda_k^+,\widetilde{\xi}_j^-)}{f(\widetilde{\xi}_j^-,\lambda_k^+)}\prod_{k=1}^{n+1}\frac{f(\widetilde{\xi}_j^-,\widetilde{\xi}_k^+)}{f(\widetilde{\xi}_k^+,\widetilde{\xi}_j^-)}\right],$$

where $|\widetilde{\xi}^-|=m-n-2$. Similarly, we obtain

$$F_{\xi_1\in\{\widetilde{\lambda}^-\}}=\sum_{n=0}^{m-1}\sum_{\substack{\{\lambda\}=\{\lambda^+\}\cup\{\lambda^-\}\\ \{\widetilde{\xi}\}=\{\widetilde{\xi}^+\}\cup\{\widetilde{\xi}^-\}\\ |\lambda^+|=|\widetilde{\xi}^+|=n}}\sum_{l=1}^{\frac{N}{2}-n}\sum_{\substack{l'=1\\ l'\neq l}}^{\frac{N}{2}-n}\frac{\mathfrak{a}(\lambda_l^-)}{\mathfrak{a}'(\lambda_l)\mathfrak{a}'(\lambda_{l'})\prod_{j=1}^{n}\mathfrak{a}'(\lambda_j^+)}$$
$$\times\frac{\overline{Y}_n(\{\lambda^+\}|\{\overline{\xi}^+\})Z_n(\{\lambda^+\}|\{\overline{\xi}\})V_n^+(\{\lambda^+\}|\{\overline{\xi}^+\})X_n(\{\lambda^+\}|\{\overline{\xi}^+\})}{\operatorname{sh}(\lambda_l^--\lambda_{l'}^-+\eta)\prod_{j=1}^{m-n-1}(1+\mathfrak{a}(\overline{\xi}_j^-))},\qquad\text{(A.4)}$$

where $(\overline{\xi}_1^+,\dots,\overline{\xi}_{n+1}^+)=(\xi_1,\widetilde{\xi}_1^+,\dots,\widetilde{\xi}_n^+)$, $\{\overline{\xi}_j^-\}_{j=1}^{m-n-1}=\{\widetilde{\xi}_j^-\}_{j=1}^{m-n-1}$, $\{\overline{\xi}\}=\{\overline{\xi}^+\}\cup\{\overline{\xi}^-\}$ and

$$
\begin{aligned}
\overline{Y}_n(\{\lambda^+\}|\{\widetilde{\xi}^+\}) &= \frac{\prod_{j=1}^n \mathfrak{b}_+(\lambda_j^+)}{\mathfrak{b}_+(\xi_1)\prod_{j=1}^n \mathfrak{b}'_+(\widetilde{\xi}_j^+)} \frac{\prod_{j=1}^n\prod_{k=1}^{n+1}[\mathrm{sh}(\lambda_j^+-\overline{\xi}_k^+-\eta)\,\mathrm{sh}(\lambda_j^+-\overline{\xi}_k^++\eta)]}{\prod_{j,k=1}^{n+1}\mathrm{sh}(\overline{\xi}_j^+-\overline{\xi}_k^++\eta)\prod_{j,k=1}^n\mathrm{sh}(\lambda_j^+-\lambda_k^+-\eta)} \\
&\quad\times \det{}_n\overline{M}_{jk}\det{}_{n+2}G(\widehat{\lambda}_j,\check{\xi}_k), \quad (\check{\xi}_1,\dots,\check{\xi}_{n+2})=(\overline{\xi}_1^+,\dots,\overline{\xi}_{n+1}^+,\xi_{m+1}), \\
\overline{M}_{jk} &= t(\widetilde{\xi}_k^+,\lambda_j^+)-t(\lambda_j^+,\widetilde{\xi}_k^+)\prod_{a=1}^n\frac{f(\lambda_a^+,\lambda_j^+)}{f(\lambda_j^+,\lambda_a^+)}\prod_{b=1}^{n+1}\frac{f(\lambda_j^+,\overline{\xi}_b^+)}{f(\overline{\xi}_b^+,\lambda_j^+)}.
\end{aligned}
$$

Now we express the sums over partitions in (A.3) and (A.4) as multiple integrals. To this end, we would like to introduce the following useful formula. Let $f(\omega_1,\dots,\omega_n)$ be a function which is analytic on and inside the contour $\mathcal{C}$, symmetric with respect to $\{\omega_j\}_{j=1}^n$, and zero when any two of its variables are the same. The poles of the function $1/(1+\mathfrak{a}(\omega))$ inside $\mathcal{C}$ are simple poles at $\omega=\lambda_j$ with residues $1/\mathfrak{a}'(\lambda_j)$, where $\{\lambda_j\}_{j=1}^{N/2}$ are the Bethe roots characterizing the largest eigenvalue of the quantum transfer matrix. Hence one has

$$
\frac{1}{n!}\int_{\mathcal{C}^n}\prod_{j=1}^n\frac{\mathrm{d}\omega_j}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}f(\omega_1,\dots,\omega_n)=\sum_{\substack{\{\lambda\}=\{\lambda^+\}\cup\{\lambda^-\}\\ |\lambda^+|=n}}\frac{f(\lambda_1^+,\dots,\lambda_n^+)}{\prod_{j=1}^n\mathfrak{a}'(\lambda_j^+)}. \tag{A.5}
$$

The relation similar to the above also holds for $\overline{\mathfrak{a}}$.

First we apply (A.5) to the partition for the set of the Bethe roots $\{\lambda\}$ in (A.3). We see that the summand in (A.3) has simple poles inside $\mathcal{C}$ at $\widehat{\lambda}_j=\widetilde{\xi}_k^+$. Since the inhomogeneous parameters $\{\xi\}$ can be chosen arbitrary values, we choose $\{\xi\}$ such that the two sets of parameters $\{\xi\}$ and $\{\lambda\}$ are distinguishable. Then there exists a simple closed contour surrounding the Bethe roots $\{\lambda\}$ but excluding $\{\xi\}$. Let $\mathcal{C}-\Gamma$ be such a contour, where Γ encircles $\{\xi\}$. Applying (A.5) into (A.3), one has

$$
\begin{aligned}
F_{\xi_1\in\{\widetilde{\lambda}^+\}} &= \sum_{n=0}^{m-2}\sum_{\substack{\{\widetilde{\xi}\}=\{\widetilde{\xi}^+\}\cup\{\widetilde{\xi}^-\}\\ |\widetilde{\xi}^+|=n+1}}\sum_{l=1}^{\frac{N}{2}-n}\sum_{\substack{l'=1\\ l'\neq l}}^{\frac{N}{2}-n}\frac{1}{n!}\int_{(\mathcal{C}-\Gamma)^n}\prod_{j=1}^n\left[\frac{\mathrm{d}\omega_j}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}\right]\frac{(-1)^n\mathfrak{a}(\lambda_l^-)}{\mathfrak{a}'(\lambda_l^-)\mathfrak{a}'(\lambda_{l'}^-)} \\
&\quad\times\frac{\widetilde{Y}_n(\{\omega\}|\{\widetilde{\xi}^+\})Z_n(\{\omega\}|\{\widetilde{\xi}\})V_n^+(\{\omega\}|\{\widetilde{\xi}^+\})X_n(\{\omega\}|\{\widetilde{\xi}^+\})}{\mathrm{sh}(\lambda_l^--\lambda_{l'}^-+\eta)(1+\mathfrak{a}(\xi_1))\prod_{j=1}^{m-n-2}(1+\mathfrak{a}(\widetilde{\xi}_j^-))}.
\end{aligned} \tag{A.6}
$$

By dividing the integrals, we transform the integrals along the contour $\mathcal{C}-\Gamma$ to those along the canonical contour $\mathcal{C}$:

$$
\int_{(\mathcal{C}-\Gamma)^n}\prod_{j=1}^n\frac{\mathrm{d}\omega_j}{2\pi\mathrm{i}}\longrightarrow\sum_{k=0}^n(-1)^k\binom{n}{k}\int_{\mathcal{C}^{n-k}}\prod_{j=1}^{n-k}\frac{\mathrm{d}\omega_j}{2\pi\mathrm{i}}\int_{\Gamma^k}\prod_{j=1}^k\frac{\mathrm{d}\omega_{n-k+j}}{2\pi\mathrm{i}}, \tag{A.7}
$$

where we have used the fact that the integrand in (A.6) is symmetric with respect to $\{\omega\}$. Noting that, inside Γ, the integrand has simple poles at $\omega_j=\widetilde{\xi}_k^+$, one can explicitly calculate

the integrals over Γ:

$$\int_{\Gamma^k}\prod_{j=1}^{k}\left[\frac{\mathrm{d}\omega_{n-k+j}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n-k+j}))}\right]\widetilde{Y}_n(\{\omega\}|\{\widetilde{\xi}^+\})Z_n(\{\omega\}|\{\widetilde{\xi}\})V_n^+(\{\omega\}|\{\widetilde{\xi}^+\})X_n(\{\omega\}|\{\widetilde{\xi}^+\})$$
$$=k!\sum_{\substack{\{\widetilde{\xi}^{++}\}\cup\{\widetilde{\xi}^{+-}\}=\{\widetilde{\xi}^+\}\\ |\widetilde{\xi}^{+-}|=k}}\widetilde{Y}_{n-k}(\{\omega_j\}_{j=1}^{n-k}|\{\widetilde{\xi}^{++}\})V_{n-k}^+(\{\omega_j\}_{j=1}^{n-k}|\{\widetilde{\xi}^{++}\})X_{n-k}(\{\omega_j\}_{j=1}^{n-k}|\{\widetilde{\xi}^{++}\})$$
$$\times\prod_{j=1}^{k}\frac{1}{1+\mathfrak{a}(\widetilde{\xi}_j^{+-})}\prod_{j=1}^{m-n-2}\left[1-\mathfrak{a}(\widetilde{\xi}_j^-)\prod_{a=1}^{n-k}\frac{f(\omega_a,\widetilde{\xi}_j^-)}{f(\widetilde{\xi}_j^-,\omega_a)}\prod_{b=1}^{n-k+1}\frac{f(\widetilde{\xi}_j^-,\widetilde{\xi}_b^{++})}{f(\widetilde{\xi}_b^{++},\widetilde{\xi}_j^-)}\right]$$
$$\times\prod_{j=1}^{k}\left[1+\prod_{a=1}^{n-k}\frac{f(\omega_a,\widetilde{\xi}_j^{+-})}{f(\widetilde{\xi}_j^{+-},\omega_a)}\prod_{b=1}^{n-k+1}\frac{f(\widetilde{\xi}_j^{+-},\widetilde{\xi}_b^{++})}{f(\widetilde{\xi}_b^{++},\widetilde{\xi}_j^{+-})}\right].\tag{A.8}$$

By inserting (A.8) via (A.7) into (A.6), the integrals on the contour $\mathcal{C}-\Gamma$ can be transformed to those on the canonical contour $\mathcal{C}$.

The remaining task is the calculation of the sums over the partition of inhomogeneous parameters $\{\xi\}$. Resumming them by using the formula as in [15]

$$\sum_{k=0}^{|x|}(-1)^k\sum_{\substack{\{x^+\}\cup\{x^-\}=\{x\}\\ |x^+|=k}}\prod_{j=1}^{|x^-|}\left[1+\kappa f(x_j^-)g(x_j^-)\right]\prod_{j=1}^{|x^+|}\left[1-\kappa g(x_j^+)\right]=\kappa^{|x|}\prod_{j=1}^{|x|}\left[g(x_j)(1+f(x_j))\right],$$

and further expressing the sum over λ_l^- (respectively $\lambda_{l'}^-$) as the integral over ω_{n+1} (respectively ω_{n+2}) by (A.5), one has

$$F_{\xi_1\in\{\widetilde{\lambda}^+\}}=\sum_{n=0}^{m-2}\sum_{\substack{\{\widetilde{\xi}\}=\{\widetilde{\xi}^+\}\cup\{\widetilde{\xi}^-\}\\ |\widetilde{\xi}^+|=n+1}}\frac{(-1)^m}{n!}\int_{\mathcal{C}^n}\prod_{j=1}^{n}\left[\frac{\mathrm{d}\omega_j\mathfrak{b}_-(\omega_j)}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}\right]\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))}$$
$$\times\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}\frac{V_n^+(\{\omega\}|\{\widetilde{\xi}^+\})W_n^-(\{\omega\}|\{\widetilde{\xi}^+\})X_n(\{\omega\}|\{\widetilde{\xi}^+\})}{\mathrm{sh}(\omega_{n+1}-\omega_{n+2}+\eta)\prod_{j=1}^{n+1}\mathfrak{b}'_-(\widetilde{\xi}_j^+)(1+\mathfrak{a}(\xi_1))}$$
$$\times\det_{n+1}M_{jk}^{(1)}(\{\omega\}|\{\widetilde{\xi}^+\})\det_{n+2}[G(\omega_j,\widetilde{\xi}_1^+),\ldots,G(\omega_j,\widetilde{\xi}_{n+1}^+),G(\omega_j,\xi_{m+1})],\tag{A.9}$$

where $W_n^-(\{\omega\}|\{\widetilde{\xi}^+\})$ and the $(n+1)\times(n+1)$ matrix $M_{jk}^{(1)}(\{\omega\}|\{\widetilde{\xi}^+\})$ are, respectively, defined by

$$W_n^\pm(\{\omega\}|\{\widetilde{\xi}^+\})=\frac{\prod_{j=1}^n\prod_{k=1}^{n+1}\mathrm{sh}(\omega_j-\widetilde{\xi}_k^+\pm\eta)\,\mathrm{sh}(\widetilde{\xi}_k^+-\omega_j\pm\eta)}{\prod_{j,k=1}^n\mathrm{sh}(\omega_j-\omega_k\pm\eta)\prod_{j,k=1}^{n+1}\mathrm{sh}(\widetilde{\xi}_j^+-\widetilde{\xi}_k^+\pm\eta)},$$
$$M_{jk}^{(1)}(\{\omega\}|\{\widetilde{\xi}^+\})$$
$$=\begin{cases}t(\omega_j,\widetilde{\xi}_k^+)+t(\widetilde{\xi}_k^+,\omega_j)\prod_{a=1}^n\frac{\mathrm{sh}(\omega_a-\omega_j-\eta)}{\mathrm{sh}(\omega_j-\omega_a-\eta)}\prod_{b=1}^{n+1}\frac{\mathrm{sh}(\omega_j-\widetilde{\xi}_b^+-\eta)}{\mathrm{sh}(\widetilde{\xi}_b^+-\omega_j-\eta)} & \text{for } j\le n\\ t(\widetilde{\xi}_k^+,\xi_1)+\mathfrak{a}(\xi_1)t(\xi_1,\widetilde{\xi}_k^+)\prod_{a=1}^n\frac{\mathrm{sh}(\omega_a-\xi_1+\eta)}{\mathrm{sh}(\omega_a-\xi_1-\eta)}\prod_{b=1}^{n+1}\frac{\mathrm{sh}(\widetilde{\xi}_b^+-\xi_1-\eta)}{\mathrm{sh}(\widetilde{\xi}_b^+-\xi_1+\eta)} & \text{for } j=n+1\end{cases}.$$

The integrand of (A.9) is a symmetric function with respect to $\{\widetilde{\xi}^+\}$ and vanishes when any two of them are the same. Thanks to this together with the fact that $1/\mathfrak{b}(\omega)$ has simple poles at $\omega=\widetilde{\xi}_k$, we can directly apply (A.5) to (A.9). Thus we arrive at

$$\begin{aligned}
F_{\xi_1\in\{\widetilde{\lambda}^+\}} &= \sum_{n=0}^{m-2}\frac{(-1)^m}{n!(n+1)!}\int_{\widetilde{\Gamma}^{n+1}}\prod_{j=1}^{n+1}\left[\frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}\mathfrak{b}_-(\zeta_j)}\right]\int_{\mathcal{C}^n}\prod_{j=1}^{n}\left[\frac{\mathrm{d}\omega_j\mathfrak{b}_-(\omega_j)}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}\right]\\
&\times\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))}\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}\frac{V_n^+(\{\omega\}|\{\zeta\})W_n^-(\{\omega\}|\{\zeta\})X_n(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1}-\omega_{n+2}+\eta)(1+\mathfrak{a}(\xi_1))}\\
&\times\det{}_{n+1}M_{jk}^{(1)}(\{\omega\}|\{\zeta\})\det{}_{n+2}[G(\omega_j,\zeta_1),\dots,G(\omega_j,\zeta_{n+1}),G(\omega_j,\xi_{m+1})], \qquad (A.10)
\end{aligned}$$

where $\widetilde{\Gamma}=\Gamma-\Gamma_{\xi_1}$; Γ_{ξ_1} surrounds the point ξ_1 but excludes $\{\widetilde{\xi}\}$.

Almost the same method is applied to $F_{\xi_1\in\{\widetilde{\lambda}^-\}}$ (A.4) by considering the integrals over the contour $\widetilde{\mathcal{C}}=\mathcal{C}-\Gamma_{\xi_1}$ instead of $\mathcal{C}$. Utilizing the transformation (A.5), and resumming the resulting equation as in the case of $F_{\xi_1\in\{\widetilde{\lambda}^+\}}$, one may have

$$\begin{aligned}
F_{\xi_1\in\{\widetilde{\lambda}^-\}} &= \sum_{n=0}^{m-1}\sum_{\substack{\{\widetilde{\xi}\}=\{\widetilde{\xi}^+\}\cup\{\widetilde{\xi}^-\}\\ |\widetilde{\xi}^+|=n}}\frac{(-1)^{m-n-1}}{n!}\int_{\widetilde{\mathcal{C}}^n}\prod_{j=1}^{n}\left[\frac{\mathrm{d}\omega_j\mathfrak{b}_-(\omega_j)}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}\right]\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))}\\
&\times\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}\frac{U_n(\{\omega\}|\{\overline{\xi}^+\})V_n^+(\{\omega\}|\{\overline{\xi}^+\})W_n^-(\{\omega\}|\{\overline{\xi}^+\})X_n(\{\omega\}|\{\overline{\xi}^+\})}{\mathrm{sh}(\omega_{n+1}-\omega_{n+2}+\eta)\mathfrak{b}_-(\xi_1)\prod_{j=1}^n\mathfrak{b}'_-(\widetilde{\xi}_j^+)}\\
&\times\det{}_n\widehat{M}_{jk}(\{\omega\}|\{\widetilde{\xi}^+\})\det{}_{n+2}[G(\omega_j,\xi_1),G(\omega_j,\widetilde{\xi}_1^+),\dots,G(\omega_j,\widetilde{\xi}_n^+),G(\omega_j,\xi_{m+1})], \qquad (A.11)
\end{aligned}$$

where the function $U_n(\{\omega\}|\{\overline{\xi}^+\})$ and $n\times n$ matrix $\widehat{M}_{jk}(\{\omega\}|\{\widetilde{\xi}^+\})$ are, respectively, written as

$$\begin{aligned}
U_n(\{\omega\}|\{\overline{\xi}^+\}) &= \prod_{j=1}^{n}\frac{\mathrm{sh}(\omega_j-\xi_1+\eta)}{\mathrm{sh}(\omega_j-\xi_1-\eta)}\prod_{k=1}^{n+1}\frac{\mathrm{sh}(\overline{\xi}_k^+-\xi_1-\eta)}{\mathrm{sh}(\overline{\xi}_k^+-\xi_1+\eta)},\\
\widehat{M}_{jk}(\{\omega\}|\{\widetilde{\xi}^+\}) &= t(\omega_j,\widetilde{\xi}_k^+)+t(\widetilde{\xi}_k^+,\omega_j)\prod_{a=1}^{n}\frac{\mathrm{sh}(\omega_a-\omega_j-\eta)}{\mathrm{sh}(\omega_j-\omega_a-\eta)}\prod_{b=1}^{n+1}\frac{\mathrm{sh}(\omega_j-\overline{\xi}_b^+-\eta)}{\mathrm{sh}(\overline{\xi}_b^+-\omega_j-\eta)}.
\end{aligned}$$

Applying again the formula (A.7) to the integration over $\widetilde{\mathcal{C}}=\mathcal{C}-\Gamma_{\xi_1}$, and noting that the sum over k in (A.7) is restricted to $k=0$ and $k=1$, we divide $F_{\xi_1\in\{\widetilde{\lambda}^-\}}$ in (A.11) into the following two parts: $F_{\xi_1\in\{\widetilde{\lambda}^-\}}=F^{(0)}_{\xi_1\in\{\widetilde{\lambda}^-\}}+F^{(1)}_{\xi_1\in\{\widetilde{\lambda}^-\}}$, where $F^{(0)}_{\xi_1\in\{\widetilde{\lambda}^-\}}$ is given by simply changing the contour $\widetilde{\mathcal{C}}\to\mathcal{C}$ in (A.11), while $F^{(1)}_{\xi_1\in\{\widetilde{\lambda}^-\}}$ is written as

$$\begin{aligned}
F^{(1)}_{\xi_1\in\{\widetilde{\lambda}^-\}} &= \sum_{n=0}^{m-2}\frac{(-1)^m}{n!(n+1)!}\int_{\widetilde{\Gamma}^{n+1}}\prod_{j=1}^{n+1}\left[\frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}\mathfrak{b}_-(\zeta_j)}\right]\int_{\mathcal{C}^n}\prod_{j=1}^{n}\left[\frac{\mathrm{d}\omega_j\mathfrak{b}_-(\omega_j)}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}\right]\\
&\times\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))}\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}\frac{V_n^+(\{\omega\}|\{\zeta\})W_n^-(\{\omega\}|\{\zeta\})X_n(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1}-\omega_{n+2}+\eta)(1+\mathfrak{a}(\xi_1))}\\
&\times\det{}_{n+1}M_{jk}^{(2)}(\{\omega\}|\{\zeta\})\det{}_{n+2}[G(\omega_j,\zeta_1),\dots,G(\omega_j,\zeta_{n+1}),G(\omega_j,\xi_{m+1})]. \qquad (A.12)
\end{aligned}$$

Note here that we have shifted the variable $n \to n+1$ and converted the sum over the partition for $\{\widetilde{\xi}\}$ into the integrals over $\widetilde{\Gamma}$. The $(n+1)\times(n+1)$ matrix $M^{(2)}_{jk}(\{\omega\}|\{\zeta\})$ is defined as

$$
\begin{aligned}
&M^{(2)}_{jk}(\{\omega\}|\{\zeta\}) \\
&\quad = \begin{cases} t(\omega_j,\zeta_k)+t(\zeta_k,\omega_j)\prod_{a=1}^{n}\frac{\mathrm{sh}(\omega_a-\omega_j-\eta)}{\mathrm{sh}(\omega_j-\omega_a-\eta)}\prod_{b=1}^{n+1}\frac{\mathrm{sh}(\omega_j-\zeta_b-\eta)}{\mathrm{sh}(\zeta_b-\omega_j-\eta)} & \text{for } j\le n \\ -t(\zeta_k,\xi_1)+t(\xi_1,\zeta_k)\prod_{a=1}^{n}\frac{\mathrm{sh}(\omega_a-\xi_1+\eta)}{\mathrm{sh}(\omega_a-\xi_1-\eta)}\prod_{b=1}^{n+1}\frac{\mathrm{sh}(\zeta_b-\xi_1-\eta)}{\mathrm{sh}(\zeta_b-\xi_1+\eta)} & \text{for } j=n+1 \end{cases}.
\end{aligned}
$$

In the next step, we would like to consider the sum $F_1 = F_{\xi_1\in\{\widetilde{\lambda}^+\}} + F_{\xi_1\in\{\widetilde{\lambda}^-\}}$ and combine the three multiple integrals into one. First we deal with the sum $F_{\xi_1\in\{\widetilde{\lambda}^+\}} + F^{(1)}_{\xi_1\in\{\widetilde{\lambda}^-\}}$. From (A.10) and (A.12), it immediately follows that

$$
\begin{aligned}
&F_{\xi_1\in\{\widetilde{\lambda}^+\}} + F^{(1)}_{\xi_1\in\{\widetilde{\lambda}^-\}} = \sum_{n=0}^{m-2}\frac{(-1)^m}{n!(n+1)!}\int_{\widetilde{\Gamma}^{n+1}}\prod_{j=1}^{n+1}\left[\frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}\mathfrak{b}_-(\zeta_j)}\right]\int_{\mathcal{C}^n}\prod_{j=1}^{n}\left[\frac{\mathrm{d}\omega_j\mathfrak{b}_-(\omega_j)}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}\right] \\
&\quad\times\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))}\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}\frac{V_n^-(\{\omega\}|\{\zeta\})W_n^-(\{\omega\}|\{\zeta\})X_n(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1}-\omega_{n+2}+\eta)} \\
&\quad\times\det_{n+1}M^-_{jk}(\{\omega\}|\{\zeta\})\det_{n+2}[G(\omega_j,\zeta_1),\dots,G(\omega_j,\zeta_{n+1}),G(\omega_j,\xi_{m+1})],
\end{aligned}
$$

where the elements of the $(n+1)\times(n+1)$ matrix $M^-(\{\omega\}|\{\zeta\})$ are given by

$$
M^-_{jk} = \begin{cases} t(\omega_j,\zeta_k)+t(\zeta_k,\omega_j)\prod_{a=1}^{n}\frac{\mathrm{sh}(\omega_a-\omega_j-\eta)}{\mathrm{sh}(\omega_j-\omega_a-\eta)}\prod_{b=1}^{n+1}\frac{\mathrm{sh}(\omega_j-\zeta_b-\eta)}{\mathrm{sh}(\zeta_b-\omega_j-\eta)} & \text{for } j\le n \\ t(\xi_1,\zeta_k) & \text{for } j=n+1 \end{cases}.
$$

Changing the contour $\widetilde{\Gamma}\to\Gamma$ and combining it with $F^{(0)}_{\xi_1\in\{\widetilde{\lambda}^-\}}$, we obtain

$$
\begin{aligned}
&F_1 = \sum_{n=0}^{m-1}\frac{(-1)^m}{n!(n+1)!}\int_{\Gamma^{n+1}}\prod_{j=1}^{n+1}\left[\frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}\mathfrak{b}_-(\zeta_j)}\right]\int_{\mathcal{C}^n}\prod_{j=1}^{n}\left[\frac{\mathrm{d}\omega_j\mathfrak{b}_-(\omega_j)}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}\right] \\
&\quad\times\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))}\int_{\mathcal{C}-\Gamma}\frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}\frac{V_n^-(\{\omega\}|\{\zeta\})W_n^-(\{\omega\}|\{\zeta\})X_n(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1}-\omega_{n+2}+\eta)} \\
&\quad\times\det_{n+1}M^-_{jk}(\{\omega\}|\{\zeta\})\det_{n+2}[G(\omega_j,\zeta_1),\dots,G(\omega_j,\zeta_{n+1}),G(\omega_j,\xi_{m+1})]. \qquad (A.13)
\end{aligned}
$$

The remaining contribution $F_2+F_3+F_4$ in (A.1) can be absorbed into (A.13) by changing the integration contours for ω_{n+1} and ω_{n+2} as $\mathcal{C}-\Gamma\to\mathcal{C}$. We thus finally arrive at

$$
\begin{aligned}
&\Phi_N(\{\xi\}) = \sum_{n=0}^{m-1}\frac{(-1)^m}{n!(n+1)!}\int_{\Gamma^{n+1}}\prod_{j=1}^{n+1}\left[\frac{\mathrm{d}\zeta_j}{2\pi\mathrm{i}\mathfrak{b}_-(\zeta_j)}\right]\int_{\mathcal{C}^n}\prod_{j=1}^{n}\left[\frac{\mathrm{d}\omega_j\mathfrak{b}_-(\omega_j)}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_j))}\right] \\
&\quad\times\int_{\mathcal{C}}\frac{\mathrm{d}\omega_{n+1}}{2\pi\mathrm{i}(1+\overline{\mathfrak{a}}(\omega_{n+1}))}\int_{\mathcal{C}}\frac{\mathrm{d}\omega_{n+2}}{2\pi\mathrm{i}(1+\mathfrak{a}(\omega_{n+2}))}\frac{V_n^-(\{\omega\}|\{\zeta\})W_n^-(\{\omega\}|\{\zeta\})X_n(\{\omega\}|\{\zeta\})}{\mathrm{sh}(\omega_{n+1}-\omega_{n+2}+\eta)} \\
&\quad\times\det_{n+1}M^-_{jk}(\{\omega\}|\{\zeta\})\det_{n+2}[G(\omega_j,\zeta_1),\dots,G(\omega_j,\zeta_{n+1}),G(\omega_j,\xi_{m+1})].
\end{aligned}
$$

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